Character rigidity for lattices and commensurators I
after Creutz-Peterson

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1 Introduction

The aim of talks C3 and C4 is to give a proof of the following theorem.

Theorem 1 ([CP13]). *Let G be a simple non-compact Lie group with property (T) and trivial centre and let H be a product of p-adic Lie groups. Let Λ be an irreducible lattice in $G \times H$.*

Then every extremal character $\tau : \Lambda \to \mathbb{C}$ is an is either almost periodic or $\tau = \delta_e$ is the left-regular character.

Here a locally compact group G has the Howe-Moore property if any unitary representation of G either has a fixed vector or it is mixing. Furthermore, a lattice $\Gamma \leq G_1 \times \cdots \times G_n$ is called irreducible, if the projections $\pi_i(\Gamma) \leq G_i$ are dense for all $i \in \{1, \ldots, n\}$. Let us mention an important example of groups satisfying the hypothesis of the previous theorem.

Example 2. The diagonal embedding of $\mathrm{PSL}(n, \mathbb{Z}[\frac{1}{p}])$ into $\mathrm{PSL}(n, \mathbb{R}) \times \mathrm{PSL}(n, \mathbb{Q}_p)$ is an irreducible lattice for all prime numbers p and all $n \geq 2$. If $n \geq 3$, then $\mathrm{PSL}(n, \mathbb{R})$ and $\mathrm{PSL}(n, \mathbb{Q}_p)$ satisfy all other hypotheses of Theorem 1.

We will deduce Theorem 1 on lattices in $G \times H$ from the next theorem about dense subgroups of G. Recall that the Schlichting completion of an inclusion $\Gamma \leq \Lambda$ by definition is the closure of the image $\Lambda \mapsto \mathrm{Sym}(\Lambda/\Gamma)$ in the topology of pointwise convergence. If Γ is commensurated by Λ, then the closure of Γ in $\Lambda//\Gamma$ is a compact open subgroup.

Theorem 3 (Theorem D of [CP13]). *Let G be a simple non-compact Lie group with property (T) and trivial centre. Assume that Λ is a countable dense subgroup of G which contains and commensurates a lattice Γ of G and such that the Schlichting completion $\Lambda//\Gamma$ is a product of topologically simple groups with the Howe-Moore property.*

Then every extremal character $\tau : \Lambda \to \mathbb{C}$ is either almost periodic or $\tau = \delta_e$.

Proof that Theorem 3 implies Theorem 1. Let $\Lambda \leq G \times H$ be an irreducible lattice as in the statement of Theorem 1. Since H is a product of p-adic Lie groups, it is totally disconnected. Hence there is a compact open subgroup $U \leq H$. Note that every compact open subgroup of H is commensurated by H. Let $\Gamma = \Lambda \cap (G \times U)$. Then Γ is commensurated by Λ. Moreover, using the fact that the projection of Λ to H is dense in H and the fact that U is compact open in H, the Schlichting completion $\Lambda//\Gamma$ satisfies

$$\Lambda//\Gamma \cong H//U \cong H \,.$$

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Denote by $\pi : G \times H \to G$ the projection onto the first factor. Since Λ is irreducible in $G \times H$ it follows that $\pi(\Lambda)$ is dense in G. Furthermore, $\pi(\Gamma)$ is discrete in G, since U is compact. It follows that $\pi(\Gamma) \leq \pi(\Lambda) \leq G$ satisfies all hypotheses of Theorem 3. To conclude the proof, we remark that by Margulis's normal subgroup theorem every normal subgroup of Γ has finite index, so that π is injective on Γ. □

Example 4. Let us continue Example 2. Consider the irreducible lattice $\Lambda = \mathrm{PSL}(n, \mathbb{Z}[\frac{1}{p}])$ in $G \times H = \mathrm{PSL}(n, \mathbb{R}) \times \mathrm{PSL}(n, \mathbb{Q}_p)$. Since $\mathrm{PSL}(n, \mathbb{Z}_p)$ is compact open in $\mathrm{PSL}(n, \mathbb{Q}_p)$, we have to consider $\Gamma = \mathrm{PSL}(n, \mathbb{Z}[\frac{1}{p}]) \cap \mathrm{PSL}(n, \mathbb{R}) \times \mathrm{PSL}(n, \mathbb{Z}_p) = \mathrm{PSL}(n, \mathbb{Z})$ in Theorem 3. Indeed, one can check that $\mathrm{PSL}(n, \mathbb{Z}[\frac{1}{p}])//\mathrm{PSL}(n, \mathbb{Z}) \cong \mathrm{PSL}(n, \mathbb{Q}_p)$ and the latter group is simple and has the Howe-Moore property.

We present now a von Neumann algebraic version of Theorem 3. This is the statement that we are going to prove. Thanks to the correspondence between extremal characters on Λ and finite factor representations of Λ, which was presented in the earlier talk C1, it suffices to show the follow theorem.

Theorem 5 (Theorem B of [CP13])**.** *Let G be a simple non-compact Lie group with property (T) and trivial centre. Assume that Λ is a countable dense subgroup of G which contains and commensurates a lattice Γ of G and such that the Schlichting completion $\Lambda//\Gamma$ is a product of topologically simple groups with the Howe-Moore property.*

If $\pi : \Lambda \to \mathcal{U}(M)$ is a finite factor representation of Λ such that $\pi(\Lambda)'' = M$, then

- *either M is finite dimensional, or*
- *π extends to an isomorphism $\mathrm{L}(\Lambda) \to M$.*

The proof of this theorem splits into two parts. In the next lecture C4, we are going to see that under the hypothesis of Theorem 5, either π extends to an isomorphism $\mathrm{L}(\Lambda) \to M$ or $\pi(\Gamma)''$ is an amenable von Neumann algebra. The latter assumption is the starting point of this lecture.

For the proof of Theorem 5 we are going to assume that $\pi(\Gamma)''$ is amenable. In particular, π cannot extend to to an isomorphism $\mathrm{L}(\Lambda) \to M$, since Γ is a property (T) group. (See Section 3).

2 Proof of the main theorem

We proceed in a direct way to the proof of Theorem 5 (under the assumption that $\pi(\Gamma)''$ is amenable). Three results are going to be used in the proof. They are further explained in Section 4 and 3, respectively.

Proposition 6. *Let Γ be discrete countable group with property (T) and let $\pi : \Gamma \to M$ be a non-degenerate representation into a finite von Neumann algebra. Then M has property (T).*

Proposition 7. *Let M be a finite von Neumann algebra which is amenable and has property (T). Then M is discrete.*

Theorem 8. *Let G be a non-discrete totally disconnected simple group with the Howe-Moore property. Then there is no non-trivial continuous homomorphism of G into the unitary group of a finite von Neumann algebra.*

We give a proof of our main Theorem 5 assuming these two results and prove them afterwards.

Proof of Theorem 5 under the assumption that $\pi(\Gamma)''$ is amenable. Let $\pi : \Lambda \to \mathcal{U}(M)$ be a finite factor representation of Λ such that $\pi(\Lambda)'' = M$ and $\pi(\Gamma)''$ is amenable. Since Γ is a lattice in a property (T) group, it has property (T) itself. By Proposition 6, it follows that $\pi(\Gamma)''$ is a finite amenable von Neumann algbera with property (T). So Proposition 7 shows that $\pi(\Gamma)''$ is completely atomic.

Consider the representation $\pi \otimes \pi^{\mathrm{op}} : \Lambda \to \mathcal{U}(\mathrm{L}^2(M) \otimes \mathrm{L}^2(M))$, which is defined by $(\pi \otimes \pi^{\mathrm{op}})(\lambda) = \pi(\lambda) \otimes J\pi(\lambda)J$. Since $\pi(\Gamma)''$ is completely atomic, there is a non-zero $(\pi \otimes \pi^{op})(\Gamma)$-invariant vector $\xi \in \mathrm{L}^2(M) \otimes \mathrm{L}^2(M)$. To see this, it suffices to check that if $e_1, \dots, e_n$ denotes an orthonormal basis of $\mathbb{C}^n$, then $\sum_{i=1}^n e_i \otimes e_i$ is invariant under all operators $U \otimes U^*$ with $U \in \mathcal{U}(n)$. Indeed,

$$\sum_{i=1}^n U e_i \otimes U^* e_i = \sum_{i=1}^n \sum_{k,l=1}^n u_{ki} \overline{u_{il}}\, e_k \otimes e_l = \sum_{k,l=1}^n \delta_{k,l}\, e_k \otimes e_l = \sum_{k=1}^n e_k \otimes e_k \,.$$

We show that ξ is $(\pi \otimes \pi^{\mathrm{op}})(\Lambda)$-invariant. Denote by p the projection onto the closed linear span of $(\pi \otimes \pi^{\mathrm{op}})(\Lambda)\xi$. Since $p(\pi \otimes \pi^{\mathrm{op}})(\Lambda)p = (\pi \otimes \pi^{\mathrm{op}})(\Lambda)p$, it follows that $p \in (\pi \otimes \pi^{\mathrm{op}})(\Lambda)'$. Define $\widetilde{M} = (\pi \otimes \pi^{\mathrm{op}})(\Lambda)''p$ and note that $\widetilde{M} \subset M \otimes M'$ is a finite von Neumann algebra. A sequence $(\lambda_n)_n$ in Λ goes to e in $\Lambda//\Gamma$ if and only if for every finite index subgroup Γ_0 of Γ there is $N \in \mathbb{N}$ such that $\lambda_n \in \Gamma_0$ for all $n \geq N$. It follows that if (λ_n) is a sequence in Λ that goes to e in $\Lambda//\Gamma$, then $(\pi \otimes \pi^{\mathrm{op}})(\lambda_n\lambda)\xi \in (\pi \otimes \pi^{\mathrm{op}})(\lambda\Gamma)\xi$ for n big enough. Consequently, $(\pi \otimes \pi^{\mathrm{op}})(\lambda_n)p \to p$ strongly. So $(\pi \otimes \pi^{\mathrm{op}})p$ extends to a strongly continuous representation of $\Lambda//\Gamma$ into the finite von Neumann algebra $\widetilde{M}$. By Theorem 8 we infer that this representation is trivial. In particular,

$$(\pi \otimes \pi^{\mathrm{op}})(\lambda)\xi = (\pi \otimes \pi^{\mathrm{op}})(\lambda)p\xi = p\xi = \xi$$

for all $\lambda \in \Lambda$. It follows that the von Neumann algebra $(\pi \otimes \pi^{\mathrm{op}})(\Lambda)''$ is not diffuse. So neither is $M = \pi(\Lambda)''$. Since M is a factor, it follows that M is finite dimensional. □

3 A rigidity vs. amenability result

In this section we show that property (T) and amenability are incompatible not only on the level of groups but also on the level of von Neumann algebras. In order to illustrate cleanly the strategy of the proof, we make use of the famous equivalence of amenability and hyperfiniteness of von Neumann algebras [Con76]. This is absolutely not necessary, but slightly simplifies the argument.

Definition 9 (Property (T) for finite von Neumann algebras)**.** Let M be a finite von Neumann algebra with faithful tracial state τ. We say that M has property (T), if the following condition holds. Whenever $(\Psi_n)_n$ is a sequence of unital completely positive and trace preserving maps such that $\|\Psi_n(x) - x\|_2 \to 0$ as $n \to \infty$ for all $x \in M$, then $\Psi_n \to \mathrm{id}$ uniformly in $\|\cdot\|_2$ on $(M)_1$.

Proposition 6 (Recall)**.** *Let Γ be discrete countable group with property (T) and let $\pi : \Gamma \to M$ be a non-degenerate representation into a finite von Neumann algebra. Then M has property (T).*

Proof. Denote by τ a faithful normal trace on M. Let $(\Psi_n)_n$ be a sequence of unital completely positive maps on M that converges to id_M pointwise in $\|\cdot\|_2$. The functions

$$\varphi_n(g) = \tau(\Psi_n(\pi(g))\pi(g)^*) = \langle \Psi_n \circ \pi(g) J\pi(g)J1, 1\rangle_{\mathrm{L}^2(M)}$$

are of positive type. Note that $\varphi_n \to 1$ pointwise. Since Γ has property (T), it follows that $\varphi_n \to 1$ uniformly. So

$$\|\Psi_n(\pi(g)) - \pi(g)\|_2^2 = \|\Psi_n(\pi(g))\|^2 + \|\pi(g)\|_2^2 - 2\mathrm{Re}\langle\Psi_n(\pi(g), \pi(g)\rangle = 2 - 2\mathrm{Re}\varphi_n(g) \to 0$$

uniformly for all $g \in \Gamma$. This implies that $\|\Psi_n(x) - x\|_2 \to 0$ uniformly on the $\|\cdot\|_2$-closure C of $\pi((\mathbb{C}\Gamma)_{\mathrm{L}^1(\Gamma),1})$. By Kaplasky's density theorem, $\bigcup_n nC = M$. So it follows that C contains a $\|\cdot\|_\infty$ open set. We infer that $(\Psi_n)_n$ converges uniformly in $\|\cdot\|_2$ to id_M on $(M)_1$. □

Proposition 7 (Recall)**.** *Let M be a finite von Neumann algebra which is amenable and has property (T). Then M is discrete.*

Proof. Assume that M is not discrete under the hypothesis of the proposition. Then there is a central projection $p \in \mathcal{Z}(M)$ such that pM is diffuse. Since pM is amenable and has property (T), we may replace 1_M by p and assume that M is diffuse. Since M is amenable, there is a sequence of finite rank completely positive unital maps $\Psi_n : M \to M$ such that $\Psi_n(x) \to x$ in $\|\cdot\|_2$ as $n \to \infty$. Since M is diffuse there is a sequence of unitaries $(u_n)_n$ in M such that $u_n \to 0$ weakly as $n \to \infty$. By property (T), there is some $N \in \mathbb{N}$ such that $\|\Psi_N(u_n) - u_n\|_2^2 < 1/2$ for all $n \in \mathbb{N}$. Denote by A the finite dimensional image of Ψ_N. Because $(u_n)_n$ goes to 0 weakly, there is some $M \in \mathbb{N}$ such that $|\tau(u_M x)| < 1/4$ for all $x \in A$. But this implies

$$\|\Psi_N(u_M) - u_M\|_2^2 = \|\Psi_N(u_M)\|_2^2 + \|u_M\|_2^2 - 2\mathrm{Re}\tau(\Psi(u_M^*)u_M) > 0 + 1 - 2\frac{1}{4} = \frac{1}{2},$$

contradicting $\|\Psi_N(u_n) - u_n\|_2^2 < 1/2$. We have shown that M cannot be diffuse, which finishes the proof. □

4 Absence of finite von Neumann algebra representations of simple Howe-Moore groups

Theorem 8 (Recall)**.** *Let G be a non-discrete totally disconnected simple group with the Howe-Moore property. Then there is no non-trivial continuous homomorphism of G into the unitary group of a finite von Neumann algebra.*

Proof. Let $\pi : G \to \mathcal{U}(M)$ be a non-degenerate representation of G into a finite von Neumann algebra. Let $z \in \mathcal{Z}(M)$ be the maximal central projection such that $\pi(G)''z = z\mathbb{C}1$. Replacing M by $(1-z)M$, we have to show that there is a non-trivial central projection in $\mathcal{Z}(M)$ under which π is trivial.

Since G is totally disconnected the space

$$\{\xi \in \mathrm{L}^2(M) \,|\, \xi \text{ is } K\text{-invariant for some } K \leq G \text{ compact open}\}$$

is dense in $\mathrm{L}^2(M)$. Hence there is a compact open subgroup $K \leq G$ and a K-invariant vector $\xi \in \mathrm{L}^2(M)$ such that $\|\xi - \hat{1}\|_2 < 1/4$. It follows that the positive type function

$$\varphi : g \mapsto \langle\pi(g)\xi, \xi\rangle$$

satisfies $|\varphi(gh) - \varphi(hg)| < 1/2$. Moreover, φ is identically 1 on K.

Since G is simple and K is commensurated by G, [BH89, Theorem 3] implies that the set $\{[K : K \cap gKg^{-1}] \mid g \in G\}$ is unbounded. If follows that $\bigcup_{g\in G} gKg^{-1}$ can not be covered by finitely many cosets of K and it is hence not compact. Since $\varphi(\bigcup_{g\in G} gKg^{-1}) \subset (1/2, 1]$, the representation π is not mixing. Using the Howe-Moore property, we infer that there is a non-zero G-fixed vector ξ for π.

Now let $p \in \mathcal{B}(\mathrm{L}^2(M))$ be the projection onto the G-invariant vectors. We already saw that $p \neq 0$. Moreover, p is invariant under multiplication by elements from $M = \pi(G)''$ and $M' = \pi(G)'$. So $p \in \mathcal{Z}(M)$. Since $\pi(g)p = p$ for all $g \in G$, we found a non-zero central projection in M under which π is trivial. This finishes the proof. □

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