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Categories of representations and classification of von Neumann algebras and quantum groups

Sven Raum

Supervisory Committee:
Prof. dr. W. Van Assche, chair
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Dissertation presented in partial fulfillment of the requirements for the degree of Doctor in Mathematics

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Preface

This thesis constitutes the ending of my four year PhD in Leuven, Belgium. It was a time full of important events for me and I want to take this opportunity to thank people who have contributed to the pleasant, instructive and interesting days I passed here in Belgium.

First of all, I want to thank Stefaan Vaes, my advisor, for his professional way of guiding me through my PhD, for what felt like having time to help at any moment I asked for it and for creating a pleasant atmosphere in our work group. I also want to thank all my coauthors who worked with me during the last four years. It was a pleasure to work and discuss with you. I learnt a lot about both mathematics and writing. In particular, I want to thank Cyril Houdayer for giving me the opportunity to work with him in Lyon next year. I am looking forward to a productive and enjoyable time. My colleagues in Leuven did their part to make the past four years an enjoyable period. Thank you very much for both the time at the department as well as the hours that we spent our free time together. Likewise, I want to thank all colleges I met during conferences in the past four years for the inspiring discussions about mathematics as well as for the good time we passed together. I want to thank my friends, especially in Münster, who stayed loyal to me although we coundn't see each other very often. I am thankful to enjoy your friendship. I want to thank all my family for offering a warm place at home whenever I come. Thank you for supporting my choice to move around the globe. Finally, I want to thank especially my brother Martin. Although we live in different places, we stay close in thought and feeling. It is unique to have someone like you - we must be grateful that live gave us the opportunity to have each other. I thank you for your support, your understanding and your honesty.

Abstract

We present several classification results and calculation of categories of representations for von Neumann algebras and quantum groups. The work is structured according to its previous publication as preprints or journal articles and grouped as two blocks, the first one dealing with quantum groups, the second one with II_1 factors. We recommend the reader who is not familiar with the subject of this work, to consult directly Chapter 1.

In the first article, the fusion rules of the category of corepresentations of several free quantum groups are calculated and we prove a theorem relating fusion rules of a free complexification of an orthogonal quantum group with the fusion rules of the original quantum group. The next article (joint work with Moritz Weber) contains classification results for easy quantum groups. We classify a large subclass of easy quantum groups in terms of reflection groups. This allows us to prove that easy quantum groups form a rich and complex object of study. The work also exhibits a fairly large class of non-classical quantum isometry groups.

In the first article on von Neumann algebras (joint work with Niels Meesschaert and Stefaan Vaes), we give a new proof for stable orbit equivalence of arbitrary Bernoulli actions of finite rank free groups - a result earlier shown by Bowen. Moreover, we can extend Bowen's work and prove orbit equivalence with some quotients of Bernoulli actions. This implies stable isomorphism of the associated group measure space II_1 factors. The second article on von Neumann algebras (joint work with Sébastien Falguières) contains our work on bimodule categories of II_1 factors. We prove that for a tensor C^*-category from a fairly large class, including finite tensor C^*-categories, there is a II_1 factor whose category of finite index bimodules is equivalent to this category. We also include consequences for the calculation of other invariants of II_1 factors. The last article contains partial classification results for free Bogoliubov crossed products by the integers. These include isomorphism as well as non-isomorphism results. We also conjecture a characterisation of strong solidity for free Bogoliubov crossed products and support it with our results.

Our work is complemented by an introduction to the history of the subject and a list of open problems illustrating the common direction of our research.

Beknopte samenvatting

Deze thesis bevat verschillende resultaten over de classificatie en de berekening van representatiecategorieën van von Neumannalgebra's en kwantumgroepen. De hoofdstukken komen overeen met eerder gepubliceerde preprints en artikels. Deze zijn in twee delen gegroepeerd. Het eerste bevat resultaten over kwantumgroepen, terwijl het tweede over von Neumannalgebra's gaat.

In het eerste artikel berekenen we de fusieregels van corepresentatiecategorieën van enkele vrije kwantumgroepen. Bovendien tonen we een verband aan tussen de fusieregels van een orthogonale kwantumgroep en van zijn vrije complexificatie. Het volgende artikel (met medeauteur Moritz Weber) bevat resultaten over de classificatie van een grote deelklasse van *easy* kwantumgroepen aan de hand van reflectiegroepen. We leiden hieruit af dat easy kwantumgroepen een rijk en complex onderwerp vormen. Bovendien vinden we een redelijk grote klasse van niet klassieke kwantumisometriegroepen.

In het eerste artikel over von Neumannalgebra's (met medeauteurs Niels Meesschaert en Stefaan Vaes) geven we een nieuw bewijs voor het feit dat vrije groepen van eindige rang vele paarsgewijs stabiel orbietequivalente acties hebben. Hun Bernoulli-acties en zekere quotiënten ervan zijn allemaal stabiel orbietequivalent. Bowen toonde al vroeger aan dat Bernoulli-acties van vrije groepen van eindige rang paarsgewijs stabiel orbietequivalent zijn. Stabiele orbietequivalentie van acties impliceert dat de bijhorende *group measure space*-constructies stabiel isomorf zijn. Het tweede artikel over von Neumannalgebra's (met medeauteur Sébastien Falguières) bevat ons werk over bimodulecategorieën. We tonen aan dat vele tensor-C^*-categoriëen als bimodulecategorie van een II_1-factor voorkomen. Onder andere realiseren we alle eindige tensor-C^*-categoriëen, wat gevolgen heeft voor andere invarianten van II_1-factoren. Het laatste artikel bevat een gedeeltelijke classificatie van gekruiste producten met vrije Bogoliubov-acties van de gehele getallen. We bewijzen isomorfisme- en niet-isomorfismeresultaten voor dergelijke von Neumannalgebra's. Verder formuleren we een conjectuur die *strong solidity* van deze von Neumann algebras zou karakteriseren, welke we met enkele van onze resultaten ondersteunen.

Onze onderzoeksresultaten worden verder aangevuld met een historische inleiding van de materie en een lijst van open problemen die de richting van ons onderzoek verduidelijken.

Contents

Chapter 1

Historical introduction and description of the main results

In this chapter, we give an introduction to our work during the last 4 years. We have two aims in this chapter. Firstly, we want to put our research into a historical context, explaining the development of the subjects it is related to. Secondly, we explain our results and the work it is based on and the motivation to do research on these topics. We give a historical introduction to the topics of von Neumann algebras in Section 1.1, (quantum) symmetries of spaces in Section 1.2, and unitary representation theory in Section 1.3. The last Section 1.4 describes our main results and indicated the common direction of our the research. We give in particular a link between our work on von Neumann algebras, measured equivalence relations and quantum groups. More on potential links between the topics we treat in this thesis can be found in Chapter 7. It is possible to directly jump to Section 1.4, skipping all historical context.

1.1 Von Neumann algebras

Von Neumann algebras were introduced by Murray and von Neumann in a series of papers starting in 1936 [139, 140, 228, 141]. As a motivation to introduce them, they name problems in operator theory, the theory of unitary group representations and quantum mechanics. A von Neumann algebra is a strongly closed, unital *-subalgebra of the algebra $\mathscr{B}(H)$ of bounded linear operators on a (complex) Hilbert space H. Von Neumann algebras have an algebraic characterisation as those unital *-subalgebras of $\mathscr{B}(H)$ that are equal to their bicommutant. If $\mathcal{S}$ denotes a subset of $\mathscr{B}(H)$, then $\mathcal{S}' = \{T \in \mathscr{B}(H) \mid ST = TS \text{ for all } S \in \mathcal{S}\}$ is its commutant. The bicommutant of $\mathcal{S}$ is $\mathcal{S}''$. Von Neumann algebras can be characterised in the more algebraic context of C*-algebras. A C*-algebra A is an normed *-algebra with isometric involution such that $\|xx^*\| = \|x\|^2$ for all $x \in A$. Sakai and Dixmier proved that a C*-algebra has a faithful representation as a von Neumann algebra if and only if it is a dual Banach space [65, 187]. If M is a von Neumann algebra, we denote by M_* its predual. The weak-* topology on a von Neumann algebra is called the σ-weak topology and it plays an important role. The fact that von Neumann algebras possess analytic and and algebraic characterisations is a strong indication that they form an interesting and relevant class of objects.

By means of a direct integral decomposition, generalising direct sums, Murray and von Neumann could reduce the study of general von Neumann algebras to those which are simple, called factors. A factor M is characterised by the fact that that it has a trivial centre $\mathbb{C}1 = \mathcal{Z}(M) = M' \cap M$.

Before we proceed, let us make the following common assumption.

Assumption. All von Neumann algebras considered in this work are supposed to act on a separable Hilbert space.

1.1.1 Types classification of factors

There is a classification of factors in three different types and several subtypes. Let us explain this classification. A projection in a von Neumann algebra M is a self-adjoint idempotent $p = p^* = p^2 \in M$. Thanks to the so called Borel functional calculus, every von Neumann algebras contains an abundance of projection. More precisely, the linear span of all projections in a von Neumann algebra forms a norm-dense set. One important point in the theory of von Neumann algebras is hence the investigation of projections and their relations to each other. Two projections $p, q \in M$ are equivalent if there is an element $v \in M$ such that $vv^* = p$ and $v^*v = q$. If $p, q \in M$ are projections, we say that p is bigger than q if $p - q$ is still a projection. In this case we write $p \geqslant q$. A projection $p \in M$ is called minimal if all projections $q \in M$, $p \geqslant q$ satisfy $q \in \{0, p\}$. A projection $p \in M$ is called finite if no projection $p \gneqq q$ is equivalent to p. Note that a minimal projection is finite. A von Neumann algebra is called finite, if all its projections are finite. Assume that M is a factor.

- M is of type I if it contains a minimal projection.
- M is of type II if it contains no minimal projection, but a finite projection.
- M is of type III if all of its non-zero projections are infinite.

If M is of type I then there is a unique $n \in \mathbb{N} \cup \{\infty\}$ and minimal projections $(p_i)_{0 \leqslant i < n}$ in M such that $\sum_{0 \leqslant i < n} p_i = 1$, where the sum denotes the strong limit of its finite partial sums. Given this cardinality, we say that M is of type I_n. If M is of type II, it either contains an infinite projection or not. In the former case, M is of type II_∞, in the latter it is of type II_1.

There is an alternative characterisation of the different types of a von Neumann algebra, that we are going to explain now. A state on a von Neumann algebra is positive linear function $\varphi : M \to \mathbb{C}$ with norm 1. More precisely, a state φ must satisfy $\phi(x^*x) \geqslant 0$ for all $x \in M$ and $\phi(1) = 1$. Weight theory generalises the notion of states on von Neumann algebras [66, 207, 202, 47]. A weight on a von Neumann algebra M is a positively homogeneous and additive function $\varphi : M_+ \to [0, +\infty]$, where $M_+ = \{x^*x \mid x \in M\}$ denotes the cone of all positive elements in M. The weight φ is

- faithful, if $\varphi(x^*x) = 0$ implies $x = 0$;
- semifinite, if the set $m_\varphi = \{x \in M \mid \varphi(x^*x) < +\infty\}$ of all φ-2-integrable elements is σ-weakly dense in M;
- normal, if for any family of pairwise orthogonal projections $(p_i)_i$ the equality $\varphi(\sum_i p_i) = \sum_i \varphi(p_i)$ holds, where $\sum_i p_i$ is to be understood as a strongly converging sum;
- tracial, if $\varphi(uxu^*) = \varphi(x)$ for all $x \in M_+$ and all unitaries $u \in \mathcal{U}(M)$.

We abbreviate a normal semifinite faithful weight as an nsff weight. Note that a state φ on M is equivalently described as a weight whose 2-integrable elements are all of M and which satisfies $\varphi(1) = 1$. Let M be a factor.

- M is of type $\mathrm{I_n}$ if and only if there is a tracial nsff weight φ on M such that $\varphi(M_+) \subset [0, +\infty]$ is a discrete set with exactly $n + 1$ elements.
- M is of type II_1 if and only if it there is a normal tracial state φ on M such that $\varphi(M_+)$ is continuous.
- M is of type II_∞ if and only if there is a tracial nsff weight φ on M such that $\varphi(M_+)$ is continuous and contains $+\infty$.

- M is of type III if there is no tracial nsff weight on M.

For simplicity we refer to a tracial state as a trace. Finiteness of von Neumann algebras can be characterised by the existence of traces. A von Neumann algebra is finite if and only if it admits a faithful normal and tracial state. Usually we denote a trace by τ and if we want to fix a certain trace on a von Neumann algebra, we speak about a tracial von Neumann algebra.

1.1.2 Construction of von Neumann algebras

If M and N are von Neumann algebras represented on Hilbert space H and K, respectively, then we can form their tensor product $M\overline{\otimes}N$ acting on $H \otimes K$. For all operators $T \in M$, $S \in N$ the tensor product $T \otimes S = (T \otimes 1)(1 \otimes S)$ is a bounded operator on $H \otimes K$. We define

$$M\overline{\otimes}N = \{T \otimes S \mid T \in M, S \in N\}'' .$$

Another important construction for von Neumann algebras is the group von Neumann algebra. Let G be a discrete group and denote by $\ell^2(G)$ the Hilbert space with a preferred orthonormal basis $(\delta_g)_{g\in G}$. The group von Neumann algebra $\mathrm{L}(G) = \lambda(G)''$ of G is the smallest von Neumann algebra containing the image of the left regular representation $\lambda : G \to \mathcal{U}(\ell^2(G))$. It is a factor if and only if G is an icc group, that is every non-trivial conjugacy class of G is infinite.

The crossed product von Neumann algebra of a discrete group acting on a von Neumann algebra always contains a copy of the group von Neumann algebra. Let $M \subset \mathscr{B}(H)$ be a von Neumann algebra and G a discrete group acting on M via *-isomorphisms (these will be automatically continuous). Then M can be represented on the Hilbert space tensor product $\ell^2(G) \otimes H \cong \ell^2(G, H)$ via the map π defined by $(\pi(x)\xi)(g) = (g \cdot x)\xi(g)$. Then $\pi(M)$ and $1 \otimes \mathrm{L}(G)$ generate a von Neumann algebra inside $\mathscr{B}(\ell^2(G) \otimes H)$, which is denoted by $M \rtimes G$. This is the crossed product of M by the action of G.

A standard Borel space is the Borel space associated with some complete metric space. A standard Borel space X together with a Borel probability measure μ on X is called a standard probability measure space. In case G acts on a standard probability measure space (X, μ) in a non-singular way, that is it preserves the measure class of μ, then G acts on $\mathrm{L}^\infty(X)$ via the almost everywhere well defined formula $g \cdot f(x) = f(g^{-1} \cdot x)$, for all $g \in G$, $f \in \mathrm{L}^\infty(X)$. We often do not mention the measure μ explicitly and write $G \curvearrowright X$. The crossed product $\mathrm{L}^\infty(X) \rtimes G$ is called the group measure space construction of $G \curvearrowright X$. A non-singular action

$G \curvearrowright X$ is called free, if the set of fixed points of every element in G is negligible. The action is ergodic, if every globally G-invariant measurable subset of X is either negligible or co-negligible. Murray and von Neumann showed that a group measure space construction of a free and ergodic non-singular action is a factor.

1.1.3 Existence of different types of factors and reduction to type II_1 factors

Murray and von Neumann were able to show that all type I_n factors are isomorphic with $\mathscr{B}(H)$, where H is separable of dimension n. Using their group measure space construction, they also gave examples of type II_1, II_∞ and III factors. The type of $\mathrm{L}^\infty(X) \rtimes G$ for a free ergodic non-singular action $G \curvearrowright (X, \mu)$ on a diffuse standard probability measure space is given by the following criterion.

- $\mathrm{L}^\infty(X) \rtimes G$ is a type II_1 factor if $G \curvearrowright X$ preserves some probability measure in the measure class of μ.
- $\mathrm{L}^\infty(X) \rtimes G$ is a type II_∞ factor if $G \curvearrowright X$ preserves an infinite measure in the measure class of μ.
- $\mathrm{L}^\infty(X) \rtimes G$ is a type III factor if $G \curvearrowright X$ admits no invariant measure equivalent to μ.

This enabled Murray and von Neumann to give examples of all types of von Neumann algebras.

Later, building on the modular theory of Tomita-Takesaki [207, 206, 202], the work of Connes on type III factors [49] showed, that they are build up out of type II factors in a sense which can be made precise. Since every II_∞ factor can be written as a tensor product of a type II_1 factor with $\mathscr{B}(\ell^2(\mathbb{N}))$, the study of von Neumann algebras can be theoretically reduced to the study of type II_1 factors.

1.1.4 Isomorphism and non-isomorphism results for von Neumann algebras

It is notoriously difficult to prove isomorphism or non-isomorphism results for von Neumann algebras. The only two non-isomorphic II_1 factors that Murray and von Neumann were able to find, are the group von Neumann algebras

$L(S_\infty)$ of the group of finite permutations of an infinite countable set and the group von Neumann algebra $L(\mathbb{F}_2)$ of the free group of rank two. In order to distinguish them they used the concept of property Γ. A II_1 factor M with trace τ has property Γ if there is a sequence of unitary elements $(u_n)_n$ in M such that $\tau(u_n) = 0$ for all $n \in \mathbb{N}$ and $[u_n, x] \to 0$ strongly for all $x \in M$. Such a sequence is called a non-trivial central sequence. It was shown that $L(S_\infty)$ has property Γ, while $L(\mathbb{F}_2)$ does not. Only 30 years later, in 1969, McDuff published a proof of the fact that there are uncountably many pairwise non-isomorphic II_1 factors [136], see also [188] for a similar result of Sakai. Note that in 1967, Powers already showed that there are uncountably many pairwise non-isomorphic factors of type III [179].

Isomorphism and non-isomorphism of von Neumann algebras can be studied in a systematic way. A surprising isomorphism between von Neumann algebras is an isomorphism that it is not coming from any classical source. An instance of such surprising isomorphisms is the uniqueness of the amenable II_1 factor explained in Section 1.1.5. Proving such isomorphism results for von Neumann algebras follows the general strategy that flexibility of a measure theoretic setting can be used to overcome difficult algebraic problems. This thought goes hand in hand with structural and rigidity results for von Neumann algebras, allowing to recover information about classical structures from the von Neumann algebraic setting. Classification results in terms of classical data are an instance of such type of results, of which the most striking one is W*-superrigidity as explained in Section 1.1.6. As we will explain there, it is naturally linked to the notions strong solidity and uniqueness of Cartan subalgebras, two notions which became focal points of research in II_1 factor theory during the last decade.

1.1.5 Surprising isomorphism results for von Neumann algebras

Isomorphism results make use of the extraordinary flexibility of von Neumann algebras. Since they are of a measure theoretic nature, cut and paste arguments with projections can give rise to unexpected isomorphism between von Neumann algebras. The first such isomorphism result is the uniqueness of the hyperfinite II_1 factor R shown by Murray and von Neumann. A von Neumann algebra M is hyperfinite, if there is an ascending sequence of finite dimensional von Neumann subalgebras $(A_n)_n$ inside M such that M is the strong closure of $\bigcup_n A_n$. It follows, for example, that all group von Neumann algebras of locally finite icc groups are isomorphic with R, which is unexpected from an algebraic point of view.

In the 70's Connes showed that hyperfiniteness is equivalent to injectivity of a von

Neumann algebra [52]. This was a major advance in the study of von Neumann algebras. In particular, his work has consequences for the study of group von Neumann algebras and group measure space constructions. A group Γ is called amenable, if there is a Γ-invariant state on $\ell^\infty(\Gamma)$. Since the group von Neumann algebra of a discrete group is injective if and only if the group is amenable, Connes's work implies that a group von Neumann algebra of a discrete group is hyperfinite if and only if the group is amenable. Also the group measure space construction associated with a probability measure preserving (pmp) action is hyperfinite if and only if the acting group is amenable. In particular, all group von Neumann algebras of icc amenable groups are isomorphic to R and so are all group measure space constructions associated with free ergodic pmp actions of amenable groups. In what follows, we will use the terms amenable von Neumann algebra and injective von Neumann algebra interchangeably.

Another source of unexpected isomorphism results for von Neumann algebras is free probability theory. Given a non-zero projection in a free group factor, $p \in \mathrm{L}(\mathbb{F}_n)$, Rădulescu [181] and Dykema [73] proved independently that the isomorphism class of the compression $p\mathrm{L}(\mathbb{F}_n)p$ does only depend on the number $t = 1 + (n-1)/\tau(p)^2$. The resulting von Neumann algebra is denoted by $\mathrm{L}(\mathbb{F}_t)$ and we call it an interpolated free group factor. Let us mention the famous free group factor isomorphism problem, asking whether $\mathrm{L}(\mathbb{F}_n)$ is isomorphic to $\mathrm{L}(\mathbb{F}_m)$ for $n \neq m$. By the work of Dykema and Rădulescu it follows that either all free group factors are pairwise non-isomorphic or all pairwise isomorphic. Dykema developed techniques based on random matrices to show that many free products and free amalgamated products of von Neumann algebras are isomorphic to interpolated free group factors. See for example [71, 76]. Alongside, the work of Shlyakhtenko on von Neumann algebras generated by operator-valued semicircular elements [224, 200, 194] forms another source of unexpected isomorphisms with free group factors.

Exploiting the isomorphism results explained before, we show in Chapter 5 that certain crossed products of free group factors by an action of the integers are also isomorphic with an interpolated free group factor. We give a more detailed introduction to our results in Section 1.4.

In Chapter 3, we give a new proof for another type of unexpected isomorphism results. Many actions of finite rank free groups are isomorphic to Bernoulli actions and the latter give rise to stably isomorphic group measure space constructions. These results where proved before by Bowen, using different methods [40, 41].

1.1.6 Structural theory and non-isomorphism results for von Neumann algebras

Invariants of II_1 factors

It is very difficult to distinguish two von Neumann algebras. This became already apparent in the work of Murray and von Neumann and is further supported by the unexpected isomorphism results described in Section 1.1.5. On the other hand, at least conjecturally, a lot of von Neumann algebras constructed from classical data should completely remember this initial data. Let us for the moment only mention Connes' conjecture, which claims that the group von Neumann algebras of two icc property (T) groups can only be isomorphic if the groups are isomorphic. Recall in this context, that a group G has property (T), if every unitary representation π of G that contains a sequence of unit vectors $(\xi_n)_n$ such that $\|\pi(g)\xi_n - \xi_n\| \to 0$ for all $g \in G$ already contains an invariant unit vector.

One possible approach to distinguishing von Neumann algebras are invariants. Murray and von Neumann introduced the fundamental group $\mathcal{F}(M)$ of a II_1 factor M. It is defined by

$$\mathcal{F}(M) = \{\tau(p)/\tau(q) \mid p, q \in M \text{ projections and } pMp \cong qMq\} \subset \mathbb{R}_{>0} .$$

Alternatively, we have

$$\mathcal{F}(M) = \{t \in \mathbb{R}_{>0} \mid M \cong M^t\} \subset \mathbb{R}_{>0} ,$$

where M^t is the amplification of M by t defined as $p(\mathrm{M}_n(\mathbb{C}) \otimes M)p$ for some $n \in \mathbb{N}$ and $p \in \mathrm{M}_n(\mathbb{C}) \otimes M$ with $(\mathrm{Tr} \otimes \tau)(p) = t$. Note that the isomorphism class of this factor does not depend on the concrete choice of n and p. Murray and von Neumann could prove that $\mathcal{F}(R) = \mathbb{R}_{>0}$, using the fact that the hyperfinite II_1 factor is unique. Using this fact, it also follows that every II_1 factor M satisfying $M \overline{\otimes} R \cong M$ has full fundamental group. Such factors are called McDuff factors in honour of her work on a characterisation of McDuff factors by means of central sequences [137], which we are going to explain in Section 1.1.6.

Only in the 70's Connes proved that there are II_1 factors which have a fundamental group not equal to $\mathbb{R}_{>0}$. Namely, he showed that every II_1 factor with property (T) has a countable fundamental group. His results did not give a concrete calculation of any fundamental group.

Only the advent of Popa's deformation/rigidity theory [165, 164, 166, 167] made the first calculation of a fundamental group not equal to $\mathbb{R}_{>0}$ possible [164]. His work is the basis of the majority of modern structural results, calculations of invariants and non-isomorphism results for von Neumann algebras. Let us explain the basic idea of deformation/rigidity theory. A deformation of a von Neumann algebra is a sequence of completely positive maps, which converges to the identity pointwise σ-weakly. There are many different sources of deformations, most notably property (H) [165] and Popa's malleable deformations [166, 167]. A deformation of a von Neumann algebra is opposed to some rigidity property of its subalgebras forcing the deformation to converge uniformly on the unit ball of such a subalgebra. The choice of a structurally relevant deformation and the identification of rigid parts of a von Neumann algebra, often allow one to identify the position of the latter in the sense of Popa's intertwining by bimodules introduced in [166]. If $A, B \subset M$ are von Neumann subalgebras of a tracial von Neumann algebra, we say that A embeds into B inside M, if there is a *-homomorphism $\phi : A \to pB^n p$, a non-zero partial isometry $v \in \mathrm{M}_{1 \times n}(\mathbb{C}) \otimes M$ such that $v\phi(x) = xv$ for all $x \in A$. We write $A \prec_M B$ in this case.

Popa's techniques were used in [166] to give examples of II_1 factors with arbitrary countable fundamental group. Later, in [176, 178], his methods were further developed, so that Popa and Vaes were able to give examples of II_1 factors with prescribed fundamental group from a big class of subgroups of $\mathbb{R}_{>0}$ containing groups of arbitrary Hausdorff dimension. However, there is no conjectural result on all possible fundamental groups of II_1 factors.

The outer automorphism group of M is defined as $\mathrm{Out}(M) = \mathrm{Aut}(M)/\mathrm{Inn}(M)$. It was a long standing open question of Connes, whether there are II_1 factors that have only inner automorphisms. This question was settled by Ioana, Peterson and Popa in [117], where they prove that actually every abelian second countable compact group can arise as the outer automorphism group of a II_1 factor. Their results are based on deformation/rigidity techniques applied to amalgamated free products. The relevant deformation for amalgamated free products is the length deformation. It can be roughly described as deforming a word the stronger the more alternating letters from the two factors of the product it has. The results of Ioana, Peterson and Popa were generalised by Vaes, and Falguières and Vaes in two directions. On the one hand, Falguières and Vaes proved in [86] that any, not necessarily abelian, second countable compact group can arise as the outer automorphism group of a II_1 factor. On the other hand, the methods of Ioana, Peterson and Popa were developed so as to control bimodule categories of II_1 factors. It is characteristic of the method employed by Ioana, Peterson and Popa, that the resulting II_1 factors are not explicit. Only the existence of a factor with a prescribed outer automorphism group is proven making use of a Baire category argument. Likewise, all results based on their methods are non-explicit.

Given two von Neumann algebras N and M, an M-N-bimodule ${}_M\mathcal{H}_N$ is a

Hilbert space $\mathcal{H}$ with a normal *-homomorphism $\pi : M \to \mathscr{B}(\mathcal{H})$ and a normal *-antihomomorphism $\rho : N \to \mathscr{B}(\mathcal{H})$ such that $\pi(x)\rho(y) = \rho(y)\pi(x)$ for all $x \in M$ and $y \in N$. Assume that M is equipped with a tracial state τ. It is known that every left M-module ${}_M\mathcal{H}$ is isomorphic to $(\ell^2(\mathbb{N}) \otimes \mathrm{L}^2(M,\tau))p$ for some projection $p \in \mathscr{B}(\ell^2(\mathbb{N}))\overline{\otimes}M$. The dimension of ${}_M\mathcal{H}$ is defined as $(\mathrm{Tr} \otimes \tau)(p)$, where Tr denotes the non-normalised trace on $\mathscr{B}(\ell^2(\mathbb{N}))$. Similarly, the dimension of a right M-module is defined. If M and N are tracial von Neumann algebras, an M-N-bimodule ${}_M\mathcal{H}_N$ has finite Jones index if $\dim_{-N}(\mathcal{H}) < +\infty$ and $\dim_{M-}(\mathcal{H}) < +\infty$. Its Jones index is defined and $\mathrm{Index}(\mathcal{H}) = \dim_{M-}(\mathcal{H}) \dim_{-N}(\mathcal{H})$. The class of all finite index M-M-bimodules together with M-M-bimodular, bounded maps as morphisms forms a category $\mathrm{Bimod}(M)$ with the follow properties.

- $\mathrm{Bimod}(M)$ is an abelian category.
- The Hom-spaces in $\mathrm{Bimod}(M)$ are Banach spaces when equipped with the operator norm.
- The adjoint of a linear operator defines a contavariant functor $* : \mathrm{Bimod}(M) \to \mathrm{Bimod}(M)$, which fixes all objects and satisfies $\|T^*T\| = \|T\|^2$ for all morphisms T.

These properties are summarised by saying that $\mathrm{Bimod}(M)$ is a C*-category. There is also a monoidal structure on $\mathrm{Bimod}(M)$ given by the Connes tensor product $\otimes_M$. Its unit is the trivial bimodule $\mathrm{L}^2(M)$. This monoidal structure enjoys the following properties.

- The functor $\otimes_M : \mathrm{Bimod}(M) \times \mathrm{Bimod}(M) \to \mathrm{Bimod}(M)$ is bi-linear.
- The associator and the unit isomorphism of $\otimes_M$ are unitary with respect to the functor $*$.

We say that $\mathrm{Bimod}(M)$ is a tensor C*-category. There is one last piece of structure on $\mathrm{Bimod}(M)$ that we want to mention.

- For every object $\mathcal{H} \in \mathrm{Bimod}(M)$ the conjugate bimodule $\overline{\mathcal{H}}$ with left and right M-action $x\overline{\xi}y = \overline{y^*\xi x^*}$ defines a conjugate for $\mathcal{H}$. That is, there are morphisms $R : \mathrm{L}^2(M) \to \overline{\mathcal{H}} \otimes_M \mathcal{H}$ and $\overline{R} : \mathrm{L}^2(M) \to \mathcal{H} \otimes_M \overline{\mathcal{H}}$ satisfying

$$\mathcal{H} \xrightarrow{\mathrm{id}\otimes R} \mathcal{H} \otimes_M \overline{\mathcal{H}} \otimes_M \mathcal{H} \xrightarrow{\overline{R}^*\otimes\mathrm{id}} \mathcal{H} = \mathrm{id}_{\mathcal{H}}$$

and

$$\overline{\mathcal{H}} \xrightarrow{\mathrm{id}\otimes\overline{R}} \overline{\mathcal{H}} \otimes_M \mathcal{H} \otimes_M \overline{\mathcal{H}} \xrightarrow{R^*\otimes\mathrm{id}} \overline{\mathcal{H}} = \mathrm{id}_{\overline{\mathcal{H}}} .$$

All the properties above, make Bimod(M) a compact tensor C*-category. The relevance of Bimod(M) stems from the fact that it encodes several other invariants of a von Neumann algebra and moreover is an interesting invariant in its own. The fundamental group and the outer automorphism group of a II_1 factor M are encoded in the bimodule category Bimod(M) as follows. The Jones index of a bimodule can be recovered from the compact tensor C*-category structure of Bimod(M). For example, if ${}_M\mathcal{H}_M$ is an irreducible bimodule and $R, \overline{R}$ are its conjugate morphisms, then $R^* \circ R$ is a element of $\mathrm{Hom}(\mathrm{L}^2(M), \mathrm{L}^2(M)) \cong \mathbb{C}$ and $\mathrm{Index}(\mathcal{H}) \cdot \mathrm{id}_{\mathrm{L}^2(M)} = R^* \circ R$. An irreducible bimodule ${}_M\mathcal{H}_M$ is called invertible if $\mathrm{Index}(\mathcal{H}) = 1$ and the isomorphism classes of invertible elements in Bimod(M) form a group with respect to $\otimes_M$. We denote this group by Grp(M). The link between the fundamental group and the outer automorphism group of the II_1 factor M is described by the following short exact sequence

$$1 \to \mathrm{Out}(M) \to \mathrm{Grp}(M) \to \mathcal{F}(M) \to 1\,,$$

where the morphism $\mathrm{Grp}(M) \to \mathcal{F}(M)$ is given by the right M-dimension. Note also that $\mathrm{Grp}(M) = \mathrm{Out}(M^\infty)$. Another invariant, which is reflected in the bimodule category is the lattice of irreducible subfactors of M (Propositon 4.4.8). A systematic theory of subfactors was initiated by Jones in [125]. The study of subfactors can be considered as an implementation of Klein's Erlangen programme for factors. Indeed, every homomorphism between factors is injective and hence the study of subfactors is the same as studying homomorphisms between factors. We say that a subfactor $N \subset M$ is irreducible, if $N' \cap M = \mathbb{C}1$. In some cases, the short exact sequence above allows one to recover Jones invariant [125]

$$\mathcal{C}(M) = \{\lambda \in \mathbb{R}_{>0} \mid \text{ there is an irreducible subfactor } N \subset M \text{ with index } \lambda\}\,,$$

if one can calculate the category of bimodules of a II_1 factor. We will do this in Chapter 4. See also Section 1.4.3 for another calculation of $\mathcal{C}(M)$.

Vaes showed in [216] that there is a II_1 factor M for which Bimod(M) contains only multiples of the trivial bimodule. It follows that every subfactor $N \subset M$ is isomorphic to $N \subset \mathrm{M}_n(\mathbb{C}) \otimes N$ for some $n \in \mathbb{N}$. This result was followed up by work of Falguières and Vaes [87], showing that the category of finite dimensional unitary representations of any compact second countable group can be realised as the bimodule category of a II_1 factor. Note that by the work of Doplicher and Roberts, these categories are exactly the symmetric compact tensor C*-categories [67]. In our work with Falguières presented in Chapter 4, we show that many other compact tensor C*-categories can arise as the bimodule category of a II_1 factor. The class we consider contains all finite tensor C*-categories, that is compact tensor C*-categories with finitely many isomorphism classes of irreducible objects, as well as the categories of finite dimensional unitary corepresentations of many discrete quantum groups, including all discrete groups. The fact that corepresentations of discrete quantum groups play a role in the study of bimodule categories of II_1 factors, is not surprising and will be explained in Section 1.2.

Falguières, Vaes and ourselves are only able to prove the existence of II_1 factors with prescribed invariants, relying on the approach and the methods of Ioana, Peterson and Popa. Based on Popa's deformation/rigidity results for crossed product von Neumann algebras, Vaes could, however, find explicit examples of II_1 factors for which he calculated the fusion rules of their bimodule category [215]. In particular, he could show that every countable group is the outer automorphism group of an explicitly described II_1 factor. Later Deprez and Vaes [64] were able to describe the complete bimodule category of explicit II_1 factors. They also obtained calculations of $\mathcal{C}(M)$, for certain II_1 factors M, proving in particular that $\mathcal{C}(M)$ can be any set of natural numbers that is closed under taking divisors and least common multiplies. In contrast to these results, our results with Falguières in Chapter 4 give an example of $\mathcal{C}(M)$ being completely calculated and containing irrational numbers. For completeness, let us also mention that Deprez continued his work on explicit examples of II_1 factors, finding concrete calculations of fundamental groups and outer automorphism groups [62] as well as endomorphism semigroups [63].

Structural results for II_1 factors

The study of property Γ by Murray and von Neumann as well as the results of McDuff on McDuff factors [137] can be considered the starting point of a structural theory of II_1 factors. In the latter work, it is proven that a II_1 factor M tensorially absorbs the hyperfinite II_1 factor, if and only if it has non-hypercentral central sequences, or expressed in a more modern language, if and only if the asymptotic centraliser $M' \cap M^\omega$ is not abelian. Here, ω denotes a non-principle ultrafilter and M^ω is the tracial ultrapower of M. This result was a important ingredient of Connes's proof of the uniqueness of the hyperfinite II_1 factor [52] several years later. Connes showed in particular, that the group von Neumann algebra $\mathrm{L}(\mathbb{F}_2)$ can be embedded into any ultrapower of the hyperfinite II_1 factor. This lead him to ask whether every II_1 factor is embeddable into an ultrapower of R, a question which became known as Connes's embedding problem and which currently attracts strong interest due to its links with several other fields of mathematics [154]. Note that the uniqueness of the ultrapower of R is equivalent to the continuum hypothesis by [88]. The work of Connes [50] also showed that a II_1 factor does not have property Γ if and only if it is full. A von Neumann algebra M is called full, if the group of its inner automorphisms is closed in the group of all automorphisms equipped with the pointwise convergence in norm on the predual M_* of M. The study of ultrapower algebras of II_1 factors remains important until the present day [114, 172]. See also [3] for a recent work about ultrapower algebras of type III von Neumann algebras including a survey on developments in this area.

In II_1 factor theory, there are several constructions of new von Neuamnn algebras out of other ones or out of classical data. Among these, we already mentioned the group von Neumann algebras, the group measure space construction and tensor products of von Neumann algebras in Section 1.1. Since all these constructions have direct implications for the structure of their output von Neumann algebras, it is natural to ask, whether a II_1 factor arises this way. More generally, one asks if any amplification of a given II_1 factor can arise by means of the above constructions. The first result in this direction was the proof of existence of a factor which is not anti-isomorphic to itself by Connes in [51]. Such a factor cannot be the amplification of any group von Neumann algebra, since the map $G \ni g \mapsto g^{-1}$ induces an anti-isomorphism of $\mathrm{L}(G)$ with itself. The first examples of II_1 factors that cannot be written as tensor products nor as a group measure space construction were the free group factors. A II_1 factor is called prime, if it cannot be written as a tensor product of two other type II_1 factors. Ge proved in [100] that the free group factors are prime. In earlier work, Voiculescu already showed that the free group factors do not contain any Cartan subalgebra [225]. A Cartan subalgebra A of a II_1 factor M is a maximally abelian subalgebra such that the group of normalising unitaries $\mathcal{N}_M(A) = \{u \in \mathcal{U}(M) \mid uAu^* = A\}$ generates M as a von Neumann algebra. If $\mathrm{L}^\infty(X) \rtimes G$ is a group measure space construction, then $\mathrm{L}^\infty(X) \subset \mathrm{L}^\infty(X) \rtimes G$ is a Cartan subalgebra. A Cartan subalgebra arising this way is called group measure space Cartan. Since every amplification of a group measure space construction contains a Cartan algebra, interpolated free group factors $\mathrm{L}(\mathbb{F}_t)$ cannot be obtained as a group measure space construction.

In the context of Popa's deformation/rigidity theory, both primeness and absence of Cartan algebras found a more systematic treatment via the notions of solidity and strong solidity, respectively. A finite von Neumann algebra M is called solid, if the relative commutant $A' \cap M$ of any diffuse von Neumann subalgebra $A \subset M$ is amenable. We call M strongly solid, if the normaliser $\mathcal{N}_M(A)''$ of any diffuse amenable von Neumann subalgebra $A \subset M$ is amenable. Note that strong solidity implies solidity. Indeed, if M is strongly solid and $A \subset M$ is diffuse, then A contains a diffuse, amenable subalgebra B. Since $A' \cap M \subset \mathcal{N}_M(B)''$, it follows that M is solid. The notion of solidity was introduced by Ozawa in [153], while strong solidity was introduced by Ozawa and Popa in [155]. It is easy to see that every non-amenable finite von Neumann algebra that is solid, must be prime. Even more is true. Any non-amenable von Neumann subalgebra of a solid von Neumann algebra is prime. This clarifies the structural importance of this notion. Similarly, no non-amenable von Neumann subalgebra of a strongly solid von Neumann algebra contains a Cartan algebra. While the original proof of Ozawa for solidity of the free group factors [153] used C*-algebraic techniques, Popa could put this result into the framework of his deformation/rigidity theory [170], which paved the way for the work of Ozawa and Popa on strong solidity of the free group factors [155]. Ozawa and Popa also show that any group measure space construction $M = \mathrm{L}^\infty(X) \rtimes \mathbb{F}_n$ of a free ergodic profinite action of a free group with finite rank has the following property. Every diffuse amenable subalgebra $A \subset M$ either has an amenable normaliser or it embeds into $\mathrm{L}^\infty(X)$ in the sense of Popa's intertwining by bimodules. The results of Ozawa and Popa were extended further to hyperbolic groups [46] by Chifan and Sinclair. This development culminated in the work of Popa and Vaes [173, 174] showing that any trace preserving action of a hyperbolic group Γ on a von Neumann algebra B gives rise to a crossed product $M = B \rtimes \Gamma$ satisfying the following dichotomy. Whenever $A \subset M$ is a diffuse subalgebra that is amenable relative to B [160, 2, 155], then either the normaliser of A is amenable relative to B or A embeds into B inside M. Note that this is a structural result of the strongest known kind, which holds for arbitrary crossed products by a trace preserving action. In particular, the result of Ozawa and Popa is extended to arbitrary free ergodic pmp actions of free groups of finite rank. In the proof of Popa and Vaes, deformation/rigidity techniques are combined with Ozawa's idea from [153] to exploit special boundary actions of hyperbolic groups. But C*-algebraic techniques are avoided. Let us mention that the results of Ozawa and Popa and Popa and Vaes have major consequences for the study of the relation between group actions and their associated group measure space constructions. We will explain the relevant term W*-superrigidity in Section 1.1.6.

Another interesting class of II_1 factors for which structural results could be obtained are so called free Bogoliubov crossed products. Shlyakhtenko's free Krieger algebras link them to finite corners of continuous cores of free Araki-Woods factors. Voiculescu introduced the free Gaussian functor [223], which associates with a real Hilbert space $H_\mathbb{R}$ a free group factor $\Gamma(H_\mathbb{R})'' \cong \mathrm{L}(\mathbb{F}_{\dim H_\mathbb{R}})$. Denote by H the complexification of $H_\mathbb{R}$ and by $\mathcal{F}(H) = \mathbb{C}\Omega \oplus \bigoplus_{n \geqslant 1} H^{\otimes n}$ the full Fock space of H. Then every vector $\xi \in H_\mathbb{R}$ defines a creation operator $l(\xi)$ on $\mathcal{F}(H)$ defined by

$$l(\xi)\Omega = \xi \quad \text{and} \quad l(\xi)\xi_1 \otimes \cdots \otimes \xi_n = \xi \otimes \xi_1 \otimes \cdots \otimes \xi_n \, .$$

Voiculescu showed that the operator $s(\xi) = (l(\xi) + l(\xi)^*)/2$ has a semicircular distribution with variance $\|\xi\|^2$. Furthermore, if $(\xi_i)_i$ is an orthogonal family of non-zero vectors in $H_\mathbb{R}$, then $(s(\xi_i))_i$ is free in the sense of free probability theory [226]. It follows that

$$\Gamma(H_{\mathbb{R}})'' = \{s(\xi) \mid \xi \in H_{\mathbb{R}}\}'' \cong \mathrm{L}(\mathbb{F}_{\dim H_{\mathbb{R}}}) .$$

Voiculescu's construction was the starting point of two different developments. On the one hand Shlyakhtenko introduced in [192] free Araki-Woods factors and on the other hand orthogonal representations were used to construct free Bogoliubov crossed products. Let us first explain free Araki-Woods factors. Given a one-parameter group $(U_t)_{t\in\mathbb{R}}$ of orthogonal transformations of a real Hilbert space $H_{\mathbb{R}}$, we can modify the scalar product on the complexification $H_{\mathbb{C}}$ of $H_{\mathbb{R}}$ in the following way. Extend U_t by linearity to $H_{\mathbb{C}}$. Then, by Stone's theorem, there is an unbounded selfadjoint and positive operator A on $H_{\mathbb{C}}$ such that $U_t = \exp(iAt)$ for all $t \in \mathbb{R}$. Shlakhtenko showed that

$$\langle \xi, \eta \rangle_U = \left\langle \frac{2}{1 + \exp(-A)} \xi, \eta \right\rangle .$$

is a well defined inner product on $H_{\mathbb{C}}$. Denoting the completion of $H_{\mathbb{C}}$ with respect to $\langle \cdot, \cdot \rangle_U$ by H, we can again consider the full Fock space $\mathcal{F}(H)$ and the left creation operators $l(\xi)$, $\xi \in H_{\mathbb{R}}$. Wrting $s(\xi) = (l(\xi) + l(\xi)^*)/2$, the von Neumann algebra $\Gamma(H_{\mathbb{R}}, (U_t))'' = \{s(\xi) \mid \xi \in H_{\mathbb{R}}\}$ is called the free Araki-Woods factor associated with $(H_{\mathbb{R}}, (U_t))$. Shlyakhtenko proved that free Araki-Woods factors are indeed factors and that they are of type III for all non-trivial one-parameter groups (U_t). A one-parameter group (U_t) is called almost periodic if the operator $\exp(A)$ defined above has pure point spectrum. Shylakhtenko classified free Araki-Woods factors associated with almost periodic one-parameter groups, showing that they are distinguished exactly by the subgroup of $\mathbb{R}_{>0}$ generated by the eigenvalues of $\exp(A)$. However, the classification of other Araki-Woods factors proved to be difficult [195], although Houdayer [108] and Houdayer and Ricard [111] could obtain structural results for all free Araki-Woods factors and their continuous cores using Popa's deformation rigidity techniques.

Voiculescu's free Gaussian functor is functorial for isometries between real Hilbert spaces. Consequently, every representation $\pi : G \to \mathcal{O}(H_{\mathbb{R}})$ of a discrete group by orthogonal transformations on a real Hilbert space, gives rise to an action of Γ on the free group factor $\Gamma(H_{\mathbb{R}})''$, which is called a free Bogoliubov action. The crossed product $\Gamma(H_{\mathbb{R}})'' \rtimes G$ is denoted by $\Gamma(H_{\mathbb{R}}, G, \pi)''$ and we call it a free Bogoliubov crossed product. Since free Bogoliubov actions form a large class of actions on free group factors, it is natural to study them. Since the word length deformation on free group factors [165] is very strong, also here Popa's deformation/rigidity theory can be used to obtain structural results, as done by Houdayer and Shlyaktenko and Houdayer [112, 106, 105].

As described in [112], the point of contact between free Bogoliubov crossed products and free Araki-Woods factors are Shlyakhtenko's free Krieger algebras

[194]. Free Krieger algebras are von Neumann algebras generated by an operator-valued semi-circular distributed element with values in a commutative von Neumann algebra. Operator valued free probability was developed by Voiculescu and Speicher in [224] and [200]. If $A \subset M$ is an inclusion of von Neumann algebras with conditional expectation $\mathrm{E} : M \to A$, and $\eta : A \to A$ is a completely positive map, then an element $X \in M$ is called A-valued semicircular with distribution η, if $\mathrm{E}(X) = 0$, $\mathrm{E}(XaX) = \eta(a)$ for all $a \in A$ and all higher A-valued moments $\mathrm{E}(Xa_1Xa_2X \cdots Xa_nX)$ of X can be described in terms of η by means of Speichers operator-valued free cumulant formalism. The free Krieger algebra generated by an A-valued semicircular element with distribution η is denoted by $\Phi(A, \eta)$. Shlyakhtenko proved in [193] that the continuous cores of free Araki-Woods factors can be represented by free Krieger factors $\Phi(\mathrm{L}^\infty(\mathbb{R}), \eta)$. At the same time, it is observed in [112], that free Bogoliubov crossed products associated with orthogonal representations of the integers are free Krieger factors $\Phi(\mathrm{L}^\infty(\mathrm{S}^1), \eta)$. This explains the interest in the special case of free Bogoliubov crossed products by the integers. In Chapter 5, we obtain isomorphism and non-isomorphism results as well as structural results for these free Bogoliubov crossed products, the aim being to give a characterisation of strong solidity for and a classification of $\Gamma(H_\mathbb{R}, \mathbb{Z}, \pi)''$ in terms of properties of π.

1.2 (Quantum) Symmetries of measure spaces

All abelian von Neumann algebras are of the form $\mathrm{L}^\infty(X)$ for a standard measure space X. This motivates the idea to consider the theory of von Neumann algebras as non-commutative measure theory. Also, von Neumann algebras are the natural framework for non-commutative integration theory [206, 207, 202, 47, 48], supporting this point of view. If $G \curvearrowright X$ is an ergodic action of a discrete group on a standard measure space, then the quotient space X/G behaves pathologically. In order to circumvent this problem, one considers the group measure space construction of the action instead. One can also consider the measurable equivalence relation $\mathcal{R}(G \curvearrowright X)$ on X described by $x \sim g \cdot x$. As Singer showed in [196], $\mathcal{R}(G \curvearrowright X)$ is an intermediate object between $G \curvearrowright X$ and $\mathrm{L}^\infty(X) \rtimes G$ in the following precise sense. Two actions $G \curvearrowright X$ and $H \curvearrowright Y$ admit an isomorphism between $\mathrm{L}^\infty(X) \rtimes G$ and $\mathrm{L}^\infty(Y) \rtimes H$ sending the Cartan algebra $\mathrm{L}^\infty(X)$ onto $\mathrm{L}^\infty(Y)$, if and only if the equivalence relations $\mathcal{R}(G \curvearrowright X)$ and $\mathcal{R}(H \curvearrowright Y)$ are isomorphic. The theory of measurable equivalence relations became an active field of research after the discovery of Singer starting with the work of Dye on orbit equivalence of free ergodic pmp action of the integers [69, 70]. See [94, 97] for recent surveys of the topic. As mentioned in Section 1.1.5 Connes proved that any free ergodic pmp action of an infinite amenable gorup gives rise to the hyperfinite II_1 factor as its group von measure space construction. Connes, Feldman and Weiss strengthened this result and proved that such an action already gives rise to the ergodic hyperfinite II_1 equivalence relation [55] - an extension of a result by Ornstein and Weiss [151]. When studying the dependence of $\mathrm{L}^\infty(X) \rtimes G$ on the action $G \curvearrowright X$, it is hence natural to split the problem into two parts. There are three natural types of equivalence for free ergodic actions. Two free ergodic pmp actions $G \curvearrowright X$ and $H \curvearrowright Y$ are

- conjugate, if there are isomorphisms $\delta : G \to H$ and $\Delta : X \to Y$ such that for all $g \in G$ and almost every $x \in X$, we have $\Delta(g \cdot x) = \delta(g) \cdot \Delta(x)$;
- orbit equivalent, if their orbit equivalence relations are isomorphic;
- W^*-equivalent, if their group measure space constructions are isomorphic.

In the context of Popa's deformation/rigidity theory, W^*-superrigid actions became a much-studied topic. A free ergodic pmp action $G \curvearrowright X$ is called W^*-superrigid if any other free ergodic pmp action that is W^*-equivalent to $G \curvearrowright X$ is already conjugate to it. By the result of Singer, this problem naturally splits into an orbit equivalence rigidity result combined with a uniqueness of group measure space Cartan algebra result. While orbit equivalence superrigidity results were known since [93], only in [177], Popa and Vaes were able to find the first W^*-superrigid group actions. Many other W^*-superrigidity results followed [110, 116, 38]. However, as explained in Section 1.1.5, there are not many isomorphism results for von Neumann algebras, including the group measure space constructions. Proving orbit equivalence of two free ergodic pmp actions is one way to establish new isomorphism results for II_1 factors. In [40] Bowen proved that all Bernoulli shifts of a fixed free group with finite rank are pairwise orbit equivalent, showing in particular that the associated group measure space constructions are isomorphic. For different ranks of the free groups, two free ergodic actions of free groups can never be orbit equivalent, as it is shown by Gaboriau's work on L^2-Betti numbers and cost of equivalence relations [95, 96]. However, in [41] Bowen shows that all Bernoulli shifts of all finitely generated free groups are pairwise stably orbit equivalent. This shows also that the associated group measure space constructions are pairwise stably isomorphic. Note that by [173], two stably W^*-equivalent actions of finitely generated free groups are stably orbit equivalent. In Section 3, we give an elementary proof of Bowen's results. Moreover, we show that many quotients of Bernoulli shifts of free products of amenable groups are conjugate to plain Bernoulli shifts.

Passing from usual to quantum symmetries of spaces, one naturally considers quantum group actions. A compact quantum group in the sense of Woronowicz

[233, 236] is a C^*-algebra A equipped with a $*$-homomorphism $\Delta : A \to A \otimes A$ such that $(\Delta \otimes \mathrm{id}) \circ \Delta = (\mathrm{id} \otimes \Delta) \circ \Delta$ and $A \otimes A = \overline{\mathrm{span}}\, \Delta(A)(A \otimes 1) = \overline{\mathrm{span}}\, \Delta(A)(1 \otimes A)$. Compact quantum groups in the operator algebraic setting are an established notion for two reasons. Firstly, a natural development of their basic theory, including a construction of a Haar state, is possible. Secondly, a characterisation of Woronowicz's compact quantum groups in terms of concrete compact tensor C^*-categories by means of a Tannaka-Krein type duality [234] gives a point of view independent of operator algebras. Kustermans and Vaes defined locally compact groups in [131]. Although they have to assume the existence of Haar weights, their definition allows for a deep and naturally developed theory [212, 213, 5, 60, 126]. In particular, many inclusions of von Neumann algebras can be described by crossed products with locally compact quantum groups [79] or more generally with measured quantum groupoids [147, 80, 77, 78]. This shows that Vaes-Kustermans locally compact quantum groups are a correct notion within the framework of operator algebras. In Chapter 4, we prove that the category of unitary corepresentations of a discrete quantum group A acting strictly outerly [213] on a II_1 factor M can be retrieved by means of bimodules in the inclusion $M \rtimes A \subset M \rtimes A \rtimes \hat{A}$. This result is most probably folklore - see [121] for related results. The correspondence between categories of corepresentations and bimodule categories is interesting in connection with the study of invariants of von Neumann algebras.

As explained in the last paragraph, actions of quantum groups can be used to obtain information about von Neumann algebras or to construct them. Vice versa, quantum group actions can be used to find new examples of quantum groups via the construction of quantum isometry groups. After work of Banica [11, 10] and Bichon [34] on quantum isometry groups of finite structures, in [102], Goswami introduced quantum isometry groups in the context of Connes's spectral triples [53, 54]. His definition was later generalised in [20]. Despite several computations of quantum isometry groups [33, 32, 20], it remained a difficult task to obtain calculations of non-classical quantum isometry groups. In Chapter 6, we show in particular that many new quantum groups can be obtained as quantum isometry groups of certain maximal group C^*-algebras.

1.3 Unitary representation theory of quantum groups and tensor categories

It is an old idea to study mathematical objects through their categories of representations. As already mentioned, a compact quantum group in the sense of Woronowicz is even completely determined by its concrete tensor C^*-category of finite dimensional unitary corepresentations. In the classical setting, Doplicher and Roberts were able to characterise the abstract compact C^*-categories that arise as categories of finite dimensional unitary representations of a compact group. They are exactly the symmetric compact tensor C^*-categories [67]. Note, however, that the symmetric structure of a compact tensor C^*-category, if it exists, does not need to be unique - even if the the tensor category is finite [82, 59, 120]. Using the fact that all unitary representations of a finite compact quantum group have integer dimensions, one can show that there are tensor C^*-categories not arising as the category of unitary representations of a compact quantum group. The question of which compact tensor C^*-categories can be obtained as categories of finite dimensional unitary corepresentations of a compact quantum group is an active field of research [159].

In [24] Banica and Speicher approached to relation between compact quantum groups and tensor C^*-categories from the categorical point of view. Making use of Speicher's formalism of crossing and non-crossing partitions [145], which proved successful in free probability theory, they define concrete tensor C^*-categories and consider the compact quantum groups associated with them by Woronowicz's Tannaka-Krein theorem. They call the quantum groups that they obtain easy quantum groups, although combinatorial quantum groups would probably be a more suitable term. Easy quantum groups are in particular quantum subgroups of Wang's universal free quantum groups $\mathrm{A_o}(n)$, hence their C^*-algebra A is generated by elements of an orthogonal matrix $u = (u_{ij}) = (u_{ij}^*) \in \mathrm{M}_n(A)$. The class of easy quantum groups is interesting for two reasons. Firstly, the successful application of combinatorial arguments involving partitions in free probability theory by the work of Speicher gave rise to connections between free probability theory and easy quantum groups [15, 129, 18, 19]. Secondly, easy quantum groups are a priory of a completely different nature than all other known classes of compact quantum groups, which potentially yields new phenomena in the study of their representation theory and the associated operator algebras. Easy quantum groups were divided by two properties. An easy quantum group is called half-liberated if its generating elements satisfy $u_{ij}u_{kl}u_{nm} = u_{nm}u_{kl}u_{ij}$. All half-liberated easy quantum groups where classified in [232]. An easy quantum group is called hyperoctahedral if its generating elements u_{ij} are partial isometries. All non-hyperoctahedral easy quantum groups were classified in [17, 232]. In Chapter 6, we introduced the notion of simplifiable easy quantum groups. A hyperoctahedral easy quantum group is simplifiable if the squares of its generating entries u_{ij}^2 are central. We showed that the class of simplifiable easy quantum groups is not amenable to classification, by giving a concrete bijection between the lattice of simplifiable easy quantum groups and lattice of reflection groups. This shows on the one hand that the class of easy quantum groups is much richer than previously expected [17] and on the other hand it gives a concrete perspective to approach problems about the representation theory of easy quantum groups for example via word counting arguments in groups.

Many results on the structure of a compact quantum group are based on the knowledge of its fusion rules, that is the fusion rules of its category of finite unitary dimensional corepresentations [8, 18, 15, 42, 91, 119]. While the fusion rules for q-deformations of classical compact Lie groups [122, 68, 186] are the same as for their classical counterparts, in other cases it is not clear how to calculate them. Banica gave a calculation of the fusion rules of Wang's and van Daele's free orthogonal and free unitary quantum groups [230, 219] in [7, 8]. His proof was of a combinatorial kind and could be adapted to other classes of quantum groups in [9, 25, 26, 13]. Denote by $\mathrm{A_s}(n)$ Wang's quantum permutation group [231]. Then a quantum group A is called free according to [13], if $\mathrm{A_u}(n) \to A \to \mathrm{A_s}(n)$ and the category of finite dimensional unitary corepresentations of A has a combinatorial description similar to that one of easy quantum groups. In Chapter 2, we used Banica's free complexification of an orthogonal quantum group [12] and some elementary isomorphism results in order to obtain more calculations of fusion rules in the class of free quantum groups.

1.4 Description of our main results

1.4.1 Isomorphisms and fusion rules of orthogonal free quantum groups and their free complexifications

This section describes our work in Chapter 2. Let us briefly recall some facts mentioned in Sections 1.3 and 1.2. In [230], Wang defined the free unitary and the free orthogonal quantum groups. The free unitary quantum group is defined as the universal C*-algebra

$$\mathrm{A_u}(n) = \mathrm{C}^*(u_{ij}, 1 \leqslant i,j \leqslant n \,|\, u = (u_{ij}) \text{ and } \overline{u} = (u_{ij}^*) \text{ are unitary}),$$

while the free orthogonal quantum group is given by

$$\mathrm{A_o}(n) = \mathrm{C}^*(u_{ij}, 1 \leqslant i,j \leqslant n \,|\, u = \overline{u} \text{ is unitary}).$$

Both are matrix quantum groups in the sense of Woronowicz [233, 235], meaning that there is a *-homomorphism $\Delta : \mathrm{A}_*(n) \to \mathrm{A}_*(n) \otimes_{\min} \mathrm{A}_*(n)$ which satisfies $\Delta(u_{ij}) = \sum_{1 \leqslant k \leqslant n} u_{ik} \otimes u_{kj}$ and that $u, \overline{u}$ are invertible. Later, in [231], Wang defined the quantum permutation group

$$\mathrm{A_s}(n) = \mathrm{C}^*(u_{ij}, 1 \leqslant i,j \leqslant n \,|\, u = \overline{u} \text{ unitary and all } u_{ij} \text{ are projections}).$$

Let us define a last quantum group, the hyperoctahedral quantum group. It is given by

$$\mathrm{A_s}(n) = \mathrm{C}^*(u_{ij}, 1 \leqslant i, j \leqslant n \,|\, u = \overline{u} \text{ unitary and all } u_{ij} \text{ are partial isometries}) .$$

The categories of finite dimensional unitary representation $\mathrm{UCorep}(\cdot)$ of these quantum groups can be described combinatorially by means of Speicher's partitions [145]. One says that the intertwiner spaces of $\mathrm{A}_*(n)$ are "spanned by partitions". This was taken as a definition in [13]: a compact matrix quantum group (CMQG) (A, u) with u of size $n \times n$ is called free, if the canonical homomorphism $\mathrm{A_u}(n) \rightarrow \mathrm{A_s}(n)$ factors through (A, u) and $\mathrm{UCorep}(A)$ is spanned by partitions. A CMQG is called orthogonal free if it is free and it is a quotient of $\mathrm{A_o}(n)$. It was shown in [24, 232], that there are exactly seven orthogonal free quantum groups.

Isomorphism classes of finite dimensional unitary corepresentations of a compact quantum group A form a based semi-ring with basis given by isomorphism classes of irreducible elements. This semi-ring is called the Grothendieck ring of $\mathrm{UCorep}(A)$ or fusion ring of A. Fusion rules play an important role when proving properties of quantum groups as described in Section 1.3. Banica calculated the fusion rules of $\mathrm{A_o}$, $\mathrm{A_u}$ and $\mathrm{A_s}$ in [7, 8, 9]. Later Banica and Vergnioux calculated the fusion rules of other free quantum groups, among them the hyperoctahedral quantum group, in [25]. They could find a common framework to explain the fusion rules of all examples known until then, which they called free fusion rings. A free fusion ring roughly is a semi-ring R whose basis is given by words with letters in some semigroup S such that the product of elements in R can be expressed in terms of the multiplication in S. Based on the work of Banica and Vergnioux, we clarify the definition of free fusion rings in Chapter 2. We say that a quantum group has free fusion rules, if its fusion ring is free. Banica and Vergnioux asked, whether all free quantum groups have free fusion rules. This motivated the calculation of the fusion rules of further examples of free quantum groups. In Chapter 2, we find a description of two of the remaining orthogonal free quantum groups in terms of the free orthogonal quantum groups and deduce that their fusion rules are not free. Note that the seventh orthogonal free quantum group was not known at the time of our work on Chapter 2, but Weber later found a description of the remaining one in the same spirit as our work does [232].

In contrast to orthogonal free quantum groups, a complete classification of all free quantum groups is out of sight. However, Banica gave a free complexification construction in [12], which makes it possible to construct a canonical unitary quantum group out of an orthogonal quantum group. If the initial quantum group was free, so will be its free complexification. In particular, as expected, the free complexification of the free orthogonal quantum group is the free unitary quantum group. We proved in Chapter 2, that if (A, u) is an orthogonal quantum group with free fusion rules satisfying $1 \notin u^{\otimes 2k+1}$ for all $k \in \mathbb{N}$, then also the fusion rules of its free complexification are free. This gives a new example of a quantum group with free fusion rules. Note that the requirement $1 \notin u^{\otimes 2k+1}$ is necessary.

Proposition 1.4.1. *Let (A, u) be an orthogonal compact matrix quantum group for which there is $k \in \mathbb{N}$ such that $1 \in u^{\otimes 2k+1}$. Denote by $(\widetilde{A}, \widetilde{u})$ the free complexification of A. Then $\widetilde{A}$ contains a group-like unitary of order two. In particular, the fusion rules of $\widetilde{A}$ are not free.*

Proof. Take $k \in \mathbb{N}$ as in the statement of the proposition. Denote by $z \in \mathrm{C}^*(\mathbb{Z}/2\mathbb{Z})$ the non-trivial group-like unitary. Recall from [12] or Chapter 2 that the fundamental corepresentation $\widetilde{u}$ of $\widehat{A}$ equals $u \cdot z$ in the universal free product $A * \mathrm{C}^*(\mathbb{Z}/2\mathbb{Z})$. Using Frobenius duality, the assumption $1 \in u^{\otimes 2k+1}$ implies, $u \in (\widetilde{u} \otimes \overline{\widetilde{u}})^{\otimes k}$. So $1 \in u \otimes u$ implies $z \in (\widetilde{u} \otimes \overline{\widetilde{u}})^{\otimes k} \otimes \widetilde{u}$. This shows that z is in the C^*-algebra $\widetilde{A}$, which is generated by the entries of $\widetilde{u}$. □

1.4.2 A connection between easy quantum groups, varieties of groups and reflection groups

We describe joint work with Moritz Weber presented in Chapter 6. Recall the definition of free quantum groups as described in Section 1.4.1. A compact matrix quantum group (A, u) is free if there is a *-homomorphism $\mathrm{A_u}(n) \to A \to \mathrm{A_s}(s)$ that maps fundamental corepresentations onto fundamental corepresentations and such that there is a combinatorial description of the category of finite dimensional unitary corepresentations $\mathrm{UCorep}(A)$ of A. Banica and Speicher realised in [24] that not the fact that a free quantum group contains $\mathrm{A_s}(s)$ is decisive, but that the combinatorial description of their corepresentation categories plays the crucial role when considering free quantum groups. They defined easy quantum groups as quotients of $\mathrm{A_o}(n)$ such that there is a combinatorial description of $\mathrm{UCorep}(A)$ in the following precise sense. A partition is an arrangement of k upper and l lower points and lines connecting them. Formally, a partition in this sense is a partition into subsets of $\{1, \cdots, k\} \sqcup \{1, \cdots, l\}$. A partition p can be represented by a diagram in the following way:

$$\begin{matrix} \cdot & \cdot & \cdot & \cdot & \dots & \cdot & \cdot \\ & & & p & & & \\ \cdot & \cdot & \cdot & \cdot & \dots & \cdot & \cdot \end{matrix}$$

k upper points and
l lower points.

Two examples of such partitions are the following diagrams.

In the first example, all four points are connected, and the partition consists only of one block. In the second example, the left upper point and the right lower point are connected, whereas neither of the two remaining points is connected to any other point. Denote by $P(k,l)$ the set of all partitions on k upper and on l lower points. Given a partition $p \in P(k,l)$ and two multi-indices $(i_1,\dots,i_k)$, $(j_1,\dots,j_l)$, we can label the diagram of p with these numbers, both the upper and the lower row labelled from left to right, and we put

$$\delta_p(i,j) = \begin{cases} 1 & \text{if } p \text{ connects only equal indices,} \\ 0 & \text{if there is a string of } p \text{ connecting unequal indices}. \end{cases}$$

For every $n \in \mathbb{N}$, there is a map $T_p : (\mathbb{C}^n)^{\otimes k} \to (\mathbb{C}^n)^{\otimes l}$ associated with p, which is given by

$$T_p(e_{i_1} \otimes \cdots \otimes e_{i_k}) = \sum_{1 \leqslant j_1,\dots,j_l \leqslant n} \delta_p(i,j) \cdot e_{j_1} \otimes \cdots \otimes e_{j_l} .$$

A compact matrix quantum group subgroup (A,u) of $\mathrm{A_o}(n)$ is called easy [24, 17], if there is a set of partitions $\mathcal{C}$ given by $D(k,l) \subset P(k,l)$, for all $k,l \in \mathbb{N}$ such that

$$\mathrm{Hom}(u^{\otimes k}, u^{\otimes l}) = \mathrm{span}\{T_p \,|\, p \in D(k,l)\} .$$

We say in this case that the intertwiner spaces of (A,u) are spanned by partitions. Banica and Speicher actually describe purely combinatorially the possible sets of partitions $\mathcal{C}$ that arise in the definition of easy quantum groups (see Chapter 6). A set of partitions is called category of partitions if it describes the category of finite dimensional unitary corepresentations of some easy quantum group.

Let us briefly mention that an easy quantum group is free in the previous sense, if and only if its category of partitions contains only non-crossing partitions in the sense of Speicher [145]. The classification of orthogonal free quantum groups mentioned in Section 1.4.1 could be extended to particular classes of easy quantum groups. Let ⊓⊓⊓ be the partition on 4 lower points that are all connected. We call an easy quantum group hyperoctahedral if its category of partitions contains ⊓⊓⊓. It was shown in [17, 232] that an easy quantum group is either hyperoctahedral or it belongs to an explicitly known family of 14 other easy quantum groups. Let ⚹ be the partition on 3 upper and 3 lower points that connects the first point in each row, with the last point of the other row and connects the two middle points as well. An easy quantum group whose category of partitions contains ⚹ is called half-liberated. In [232], it was shown that every half-liberated hyperoctahedral quantum group is contained in a countable family of easy quantum groups defined by Banica, Curran and Speicher in [17]. This family is called the hyperoctahedral series. So it remained to describe non-half-liberated hyperoctahedral easy quantum groups.

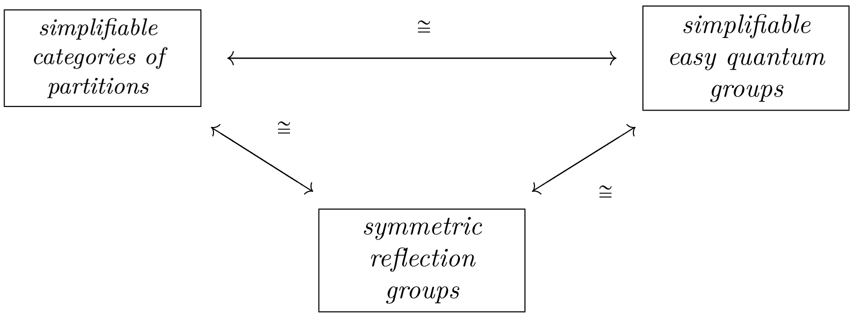
In [17], also another countable family of hyperoctahedral quantum groups was defined, which is called the higher hyperoctahedral series. The partition ⊔⁄⊓⊓ consisting of a block with four elements and a block with two elements was the common partition that is contained in all categories of partitions of elements of the higher hyperoctahedral series. We say that an easy quantum group is simplifiable, if it its category of partitions contains ⊔⁄⊓⊓. Any simplifiable easy quantum group is hyperoctahedral. In Chapter 6, we give a complete description of all simplifiable easy quantum groups, show that there are uncountably many and exploit our description in order to obtain structural results on the lattice of simplifiable easy quantum groups. We describe the main result of this chapter in what follows.

Recall that a reflection group is a countable discrete group G together with a (possibly countably infinite) family of generators $(g_i)_i$ of order two. Equivalently, one can consider normal subgroups of the infinite free product $\mathbb{Z}_2^{*\infty}$ of the group of order two. Denote by S_0 the endomorphism subsemigroup of $\operatorname{End}(\mathbb{Z}_2^{*\infty})$ generated by all inner automorphisms and the following endomorphisms called identification of letters. For all $n \in \mathbb{N}$ and all choices of $i_1, \ldots, i_n$ and $j_1, \ldots, j_n$ of indices, the map

$$\mathbb{Z}_2^{*\infty} \to \mathbb{Z}_2^{*\infty} : \begin{cases} a_{i_k} \mapsto a_{j_k} & \text{for } k = 1, \ldots, n \\ a_i \mapsto a_i & \text{if } i \notin \{i_1, \ldots, i_n\} \, . \end{cases}$$

lies in S_0. We call a reflection group G symmetric if its associated normal subgroup $H \leqslant \mathbb{Z}_2^{*\infty}$ is S_0-invariant, that is $\phi(H) \subset H$ for all $\phi \in S_0$.

Theorem (See Section 6.6.1)**.** *There is a commuting diagram of lattice isomorphisms and anti-isomorphisms*



in which all maps are explicitly described.

The explicit maps between the different lattices in the last theorem can be found in Chapter 6. We want to point out that the correspondence between simplifiable categories of partitions and simplifiable easy quantum groups can be made more explicit than in the general framework of Banica and Speicher [24] using Woronowicz's Tannaka-Krein theorem [234]. Indeed the relation between the two is rather involved, making use of the tensor C*-category associated with a category of partitions. That is why we want to highlight the following rephrasing of Proposition 6.5.5. If p is a partition with k blocks and $a_1, \dots, a_k$ are elements of an C*-algebra A, we denote by $p(a_1, \dots, a_k)$ the element of A that is obtained by labelling the blocks of p with $a_1, \dots, a_k$ clockwise starting from the top left corner and multiplying the resulting word afterwards.

Proposition (See Proposition 6.5.5)**.** *The map between simplifiable categories of partitions and simplifiable easy quantum groups can explicitly described. If $\mathcal{C}$ is a simplifiable category of partitions, then*

$$\begin{aligned}\mathrm{A}_{\mathcal{C}}(n) = C^*\Big(u_{ij}, 1 \leqslant i,j \leqslant n \,|\, u = \overline{u} \text{ unitary, } u_{ij}^2 \text{ central and} \\ p(a_1, \dots, a_k) = a_1^2 \cdots a_k^2 \text{ for all} \\ \text{choices } a_r \in \{u_{ij} \mid i,j = 1, \dots, n\}, 1 \leqslant r \leqslant k, \\ \text{and all partitions } p \in \mathcal{C}\Big)\,.\end{aligned}$$

is the associated simplifiable easy quantum group. If (A, u) is a simplifiable easy quantum group, then its associated category of partitions is

$$\begin{aligned}\mathcal{C}_A = \{p \text{ partition with } k \text{ blocks} \,|\, k \in \mathbb{N} \text{ and } p(a_1, \dots, a_k) = p(a_1^2, \dots, a_k^2) \\ \text{for all choices } a_r \in \{u_{ij} \mid i,j = 1, \dots, n\}, 1 \leqslant r \leqslant k\}\,.\end{aligned}$$

Applying our main theorem, we obtain the following result on the complexity of easy quantum groups.

Theorem (See Theorem 6.B)**.** *There is an injection of lattices of varieties of groups into the lattice of easy quantum groups. In particular, there are uncountably many easy quantum groups that are pairwise non-isomorphic.*

This result has two interpretations. On the one hand, it shows that the class of easy quantum groups is very rich. On the other hand, it says that easy quantum groups are too complex to study them all at the same time. This implies that the strategy for research on easy quantum groups has to focus on particularly interesting and accessible subclasses.

In Section 1.2, we explained that quantum isometry groups are non-classical replacements of isometry groups in the context of operator algebras. Banica and Skalski [22, 21] considered for the first time quantum isometry groups C*-algebras. We give a description of the maps between easy quantum groups and symmetric reflection groups in our main theorem in terms of such quantum isometry group constructions. Denote by $\mathcal{C}(H_n^{[\infty]})$ the maximal simplifiable easy quantum group - its category of partitions is generated by the element $\sqcup\!\!/\!\!\sqcap$. Let $E \leqslant \mathbb{Z}_2^{*\infty}$ be the subgroup of all words of even length and for $H \leqslant \mathbb{Z}_2^{*\infty}$ write $(H)_n = \mathbb{Z}_2^{*n}$.

Theorem (See Theorems 6.C)**.** *If $H \leqslant E \leqslant \mathbb{Z}_2^{*\infty}$ is a proper S_0-invariant subgroup of E, then the maximal quantum subgroup of $\mathcal{C}(H_n^{[\infty]})$ acting faithfully by isometries on $C^*(\mathbb{Z}_2^{*n}/(H)_n)$ is a simplifiable easy quantum group.*

Vice versa, the diagonal subgroup of any simplifiable easy quantum group is of the form $\mathbb{Z}_2^{*n}/(H)_n$ for some proper S_0-invariant subgroup $H \leqslant E$. Moreover, these two operations are inverse to each other.

In particular, the last theorem gives a fairly large class of new examples of non-classical quantum isometry groups.

1.4.3 Tensor C*-categories arising as bimodule categories of II_1 factors

This section explains our joint work with Sébastien Falguières presented in Chapter 4. In Section 1.1.6, we explained the history of invariants for II_1 factors. Recall that a compact tensor C*-category is called finite if it has a finite number of isomorphism classes of irreducible objects. Let us right away state two of the consequences of our work presented in Chapter 4.

Theorem (See Theorem 4.A). *For every finite tensor C*-category $\mathcal{C}$, there is a* II_1 *factor* M *such that* $\mathrm{Bimod}(M) \simeq \mathcal{C}$ *as tensor C*-categories.*

Theorem (See Corollary 4.4.4). *For every second countable compact group* G *there is a* II_1 *factor* M *such that* $\mathrm{Out}(M) \cong G$ *and every finite index bimodule of* M *is of the form* ${}_{\alpha(M)}\mathrm{L}^2(M)_M$ *for some* $\alpha \in \mathrm{Aut}(M)$*.*

The first theorem complements the work of Vaes and Falguières on bimodule categories of II_1 factors [87], where it was shown that every symmetric compact tensor C*-category is the bimodule category of a II_1 factor. The second result generalises the work of Falguières and Vaes on outer automorphism groups of II_1 factors [86], saying that every second countable compact group is the outer automorphism of a II_1 factor. In particular, it gives the first example of a completely calculated, uncountable bimodule category of a II_1 factor.

We show how to calculate Jones invariant

$$\mathcal{C}(M) = \{\lambda \mid \text{there is a finite index irreducible inclusion } N \subset M \text{ of index } \lambda\}$$

for II_1 factors with finite bimodule category. We calculate $\mathcal{C}(M)$ in a special case where it contains irrational numbers. More concretely, we prove the following theorem.

Theorem (See Theorem 4.B). *There exists a* II_1 *factor* M *such that*

$$\mathcal{C}(M) = \Big\{1, \frac{5+\sqrt{13}}{2}, 12+3\sqrt{13}, 4+\sqrt{13}, \frac{11+3\sqrt{13}}{2}, \frac{13+3\sqrt{13}}{2}, \frac{19+5\sqrt{13}}{2}, \frac{7+\sqrt{13}}{2}\Big\}.$$

All theorems in Chapter 4 are derived from a main result, which we are going to explain in what follows. An inclusion of tracial von Neumann algebras $N \subset M$ is called quasi-regular if the N-N bimodule ${}_N\mathrm{L}^2(M)_N$ is a direct sum of finite index N-N-bimodules. Note that this is equivalent to the common definition given in Chapter 4 by Section 1.4.2 in [164]. Recall that the basic construction of $N \subset M$ is the semifinite von Neumann algebra $\langle M, e_N\rangle$ acting on $\mathrm{L}^2(M)$, where $e_N : \mathrm{L}^2(M) \to \mathrm{L}^2(N) \subset \mathrm{L}^2(M)$ is the orthogonal projection. The inclusion $N \subset M$ has depth 2, if ${}_N\mathrm{L}^2(M_1)_M$ is isomorphic to a subbimodule of ${}_N\mathrm{L}^2(M)^{\oplus\infty}{}_M$. Denote by $\mathrm{Bimod}(M \subset M_1)$ the tensor C*-category in $\mathrm{Bimod}(M)$ that is generated by ${}_M\mathrm{L}^2(M_1)_M$. If $N \subset M$ has depth 2, every irreducible bimodule in $\mathrm{Bimod}(M \subset M_1)$ is isomorphic to a subbimodule of ${}_M\mathrm{L}^2(M_1)_M$.

Theorem (See Theorem 4.D). *let* $N \subset Q$ *be a quasi-regular and depth 2 inclusion of* II_1 *factors. Assume that* N *and* $N' \cap Q$ *are hyperfinite and denote* *by $N \subset Q \subset Q_1$ the basic construction. Then, there exist uncountably many pairwise non-stably isomorphic* II_1 *factors* (M_i) *such that for all i we have* $\mathrm{Bimod}(M_i) \simeq \mathrm{Bimod}(Q \subset Q_1)$ *as tensor C*-categories.*

The proof of this result heavily relies on the deformation/rigidity results of Ioana, Petersen and Popa [117]. As we explained in the introduction, the method of Ioana, Peterson and Popa is intrinsically non-constructive, as it relies on a Baire category argument. So we only show the existence of II_1 factors with prescribed bimodule category in the above theorem and don't give a concrete example. To finish this introduction, let us give a consequence of our main theorem that we did not mention in this generality yet. For a locally compact group G denote by $\mathrm{URep}_{\mathrm{fin}}(G)$ the compact tensor C*-category of finite dimensional unitary representations of G. If A is a Kac algebra, we denote similarly by $\mathrm{UCorep}_{\mathrm{fin}}(A)$ the category of finite dimensional unitary corepresentations of A. In [198], the notion of maximally almost periodic discrete Kac algebras was introduced. Roughly speaking, a Kac algebra A is maximally almost periodic, if matrix coefficients of its finite dimensional unitary corepresentations span A σ-weak densely.

Theorem (See Theorem 4.C)**.** *Let $\mathcal{C}$ denote one of the following compact tensor C*-categories. Either* $\mathcal{C} = \mathrm{URep}_{\mathrm{fin}}(G)$ *for a countable discrete group, or* $\mathcal{C} = \mathrm{UCorep}_{\mathrm{fin}}(A)$ *for an amenable or a maximally almost periodic discrete Kac algebra A. Then, there is a* II_1 *factor M such that* $\mathrm{Bimod}(M) \simeq \mathcal{C}$.

This theorem gives a concrete motivation to study discrete Kac algebras and their representation theory. Our work on free and on easy quantum groups in the Chapters 2 and 6 is very much related to this. Let us explicitly mention the question of whether the dual of the free orthogonal quantum group is maximally almost periodic, which is described in Section 7.2.2.

1.4.4 Stable orbit equivalence of Bernoulli actions of free groups and isomorphism of some of their factor actions

This Section describes our joint work with Niels Meesschaert and Stefaan Vaes, which we present in Chapter 3. If Γ is a countable infinite discrete group and (X_0, μ_0) is some standard probability measure space, the Bernoulli action of Γ with base space (X_0, μ_0) is the natural shift action on the product space $(X_0^\Gamma, \mu_0^{\otimes\Gamma})$. Extending earlier work of Ornstein on Bernoulli actions of $\mathbb{Z}$ [149, 150], Ornstein and Weiss proved that entropy is a complete invariant for isomorphism of Bernoulli actions of any discrete amenable group [152]. Moreover, they could show that certain factor actions of Bernoulli actions are isomorphic with a Bernoulli shift. This is interesting, since due to the characterisation by entropy, Bernoulli shifts are comparably well understood. As it was proved by Connes, Feldman and Weiss in [55], all free ergodic pmp actions of amenable groups are pairwise orbit equivalent, which makes them also indistinguishable on the level of their group measure space constructions.

In [39], Bowen introduced a generalisation of entropy for actions of amenable groups and he showed that Bernoulli actions of a free group with different base space entropy cannot be isomorphic. This made the question of whether such Bernoulli actions can be orbit equivalent particularly interesting. In [40], Bowen showed that indeed all non-trivial Bernoulli shifts of a fixed finite rank free group are pairwise orbit equivalent. In [41], he showed that Bernoulli shifts of free groups with different rank are stably orbit equivalent. Note that such actions cannot be orbit equivalent due to the work of Gaboriau on L^2-Betti numbers and cost of measured equivalence relations [95, 96]. The proofs of Bowen were graph theoretical in nature. We gave new proofs of Bowen's results using elementary algebraic methods only relying on the universal property of the free groups and an abstract characterisation of Bernoulli shifts and of co-induced actions. This gives the following theorem.

Theorem (Bowen [41, 40]. See Theorem 3.A)**.** *For fixed n and varying non-trivial base probability space (X_0, μ_0) the Bernoulli actions $\mathbb{F}_n \curvearrowright X_0^{\mathbb{F}_n}$ are orbit equivalent.*

If also n varies, the Bernoulli actions $\mathbb{F}_n \curvearrowright X_0^{\mathbb{F}_n}$ and $\mathbb{F}_m \curvearrowright Y_0^{\mathbb{F}_m}$ are stably orbit equivalent with compression constant $(n-1)/(m-1)$.

The abstract characterisation of Bernoulli shifts and co-induced actions actually allows us to identify factor actions of finite free products of infinite amenable groups $\Gamma = \Lambda_1 * \cdots * \Lambda_n \curvearrowright K^\Gamma/K$ for any compact second countable group K as Bernoulli shifts.

Theorem (See Theorem 3.B)**.** *If $\Gamma = \Lambda_1 * \cdots * \Lambda_n$ is the free product of n infinite amenable groups and if K is a non-trivial second countable compact group equipped with its normalized Haar measure, then the factor action $\Gamma \curvearrowright K^\Gamma/K$ of the Bernoulli action $\Gamma \curvearrowright K^\Gamma$ by the diagonal translation action of K is isomorphic with a Bernoulli action of Γ. In particular, keeping n fixed and varying the Λ_i and K, all the actions $\Gamma \curvearrowright K^\Gamma/K$ are orbit equivalent.*

In the particular case where $\Gamma = \mathbb{F}_n$, the action $\Gamma \curvearrowright K^\Gamma/K$ is isomorphic with the Bernoulli action $\Gamma \curvearrowright (K \times \cdots \times K)^\Gamma$ whose base space is an n-fold direct product of copies of K.

This extends results of Ornstein and Weiss [152]. It was speculated before whether the actions $\Gamma \curvearrowright K^\Gamma/K$ for fixed non-amenable Γ and for K running through compact groups, gives rise to explicit examples of pairwise non-orbit equivalent actions. Out results show that this is not the case.

The above results, from a von Neumann algebraic perspective, give a complete classification of an interesting and natural class of group measure space constructions $\mathrm{L}^\infty(K^{\mathbb{F}_n}/K) \rtimes \mathbb{F}_n$ and $\mathrm{L}^\infty(X_0^{\mathbb{F}_n}) \rtimes \mathbb{F}_n$, saying that any of these factors are stably isomorphic with scaling factor $(m-1)/(n-1)$ if the free groups acting have m and n generators, respectively. The classification involves unexpected isomorphism results for von Neumann algebras, as they were explained in Section 1.1.5.

1.4.5 On the classification of free Bogoliubov crossed product von Neumann algebras by the integers

We describe our work presented in Chapter 5. There are different ways to associate an action $G \curvearrowright M$ on a tracial von Neumann algebra with an orthogonal representation of a discrete group G. Namely, there are Gaussian actions, Bogoliubov actions and free Bogoliubov actions on the diffuse abelian von Neumann algebra, the hyperfinite II_1 factor and on free group factors, respectively. They have in common that the original representation of G is closely related to the representation of G on the associated L^2-space of the von Neumann algebras. It is, however, not clear how the crossed product von Neumann algebra $M \rtimes G$ is related to $G \curvearrowright M$. For Gaussian actions, the focus was put on rigidity results involving only groups that have a certain rigidity property themselves (that is property (T) groups and non-amenable products of infinite groups) [38] and for Bogoliubov actions research focused on entropy results for abelian groups [201, 30, 101]. In contrast, for free Bogoliubov actions the assumptions on G are of a more general nature and the questions one asks become different. Let us denote a free Bogoliubov action associated with an orthogonal representation (H, π) of G by $G \curvearrowright \Gamma(H)''$ and the free Bogoliubov crossed product by $\Gamma(H, G, \pi)''$. On the one hand, Houdayer and Shlyakhtenko [112] and Houdayer [106] proved strong structural results about free Bogoliubov crossed product of any countable group, making only assumptions on the representation they are constructed from. On the other hand, Houdayer could prove maximal amenability of $\mathrm{L}G \subset \Gamma(H)'' \rtimes G$ for any weakly mixing representation of an infinite abelian discrete group G. In particular, also abelian groups give rise to interesting free Bogoliubov crossed products, in contrast to the case of Gaussian actions and Bogoliubov actions. The simplest group is supposedly the group of integers, which motivates to study it in a deeper way.

The work in Chapter 5 studies free Bogoliubov crossed products with $\mathbb{Z}$ and aims at a classification and a characterisation of structural properties of these von Neumann algebras in terms of properties of the representation they are constructed from. Concerning the classification of free Bogoliubov crossed products with $\mathbb{Z}$, we can divide our results into three types. Firstly, we obtain a complete classification of free Bogoliubov crossed products associated with periodic orthogonal representations of $\mathbb{Z}$. Notably, the classification of such crossed products is equivalent to the free group factor isomorphism problem.

Theorem (See Theorem 5.A). *Let (π, H) be a non-faithful orthogonal representation of $\mathbb{Z}$ of dimension at least 2. Let $r = 1 + (\dim \pi - 1)/[\mathbb{Z} : \ker \pi]$. Then*

$$\Gamma(H, \mathbb{Z}, \pi)'' \cong \mathrm{L}^\infty([0,1]) \overline{\otimes} \mathrm{L}\mathbb{F}_r\,,$$

by an isomorphism carrying the subalgebra $\mathrm{L}\mathbb{Z}$ of $\Gamma(H, \mathbb{Z}, \pi)''$ onto the subalgebra $\mathrm{L}^\infty([0,1]) \otimes \mathbb{C}^{[\mathbb{Z}:\ker \pi]}$ of $\mathrm{L}^\infty([0,1]) \overline{\otimes} \mathrm{L}\mathbb{F}_r$

The second type of classification result that we obtain are flexibility results, mainly focusing on almost periodic representations of $\mathbb{Z}$. Let us first note the following theorem.

Theorem (See Theorem 5.B). *The isomorphism class of the free Bogoljubov crossed product associated with an orthogonal representation π of $\mathbb{Z}$ with almost periodic part π_{ap} depends at most on the weakly mixing part of π, the dimension of π_{ap} and the concrete embedding into S^1 of the group generated by the eigenvalues of the complexification of π_{ap}.*

Shlyakhtenko asked during the conference on von Neumann algebras and ergodic theory in IHP, Paris, 2011, whether two free Bogoliubov crossed products associated with almost periodic orthogonal representations of the integers are isomorphic if and only if the concrete subgroup of S^1 generated by the eigenvalues of the complexifications of the representations they are constructed from are the same. We could answer this question in the negative.

Theorem (See Theorem 5.D). *All faithful two dimensional representations of $\mathbb{Z}$ give rise to isomorphic free Bogoljubov crossed products.*

It remains open to find a complete classification of free Bogoliubov crossed products associated with almost periodic orthogonal representation of the integers. As we point out, our work allows to single out a conjecture on how a complete classification should look like.

Conjecture (See Conjecture 5.A). *The abstract isomorphism class of the subgroup generated by the eigenvalues of the complexification of an infinite dimensional, faithful, almost periodic orthogonal representation of $\mathbb{Z}$ is a* *complete invariant for isomorphism of the associated free Bogoljubov crossed product.*

In order to show that unexpected isomorphisms for free Bogoliubov crossed products exist, we prove the following theorem.

Theorem (See Theorem 5.C)**.** *If λ denotes the left regular orthogonal representation of $\mathbb{Z}$ and $\mathbb{1}$ denotes its trivial representation, then*

$$\Gamma(\ell^2(\mathbb{Z}) \oplus \mathbb{C}, \mathbb{Z}, \lambda \oplus \mathbb{1})'' \cong \Gamma(\ell^2(\mathbb{Z}), \mathbb{Z}, \lambda)'' \not\cong \Gamma(\ell^2(\mathbb{Z}) \oplus \mathbb{C}^2, \mathbb{Z}, \lambda \oplus 2 \cdot \mathbb{1})'' .$$

The third type of classification result we obtain are rigidity results for representations containing a two-dimensional invariant subspace. We are able to recover spectral information of the involved representations. This leads to a number of non-isomorphism results for free Bogoliubov crossed products, which are summarised in the following theorem.

Theorem (See Theorem 5.G)**.** *No free Bogoliubov crossed product associated with a representation in the following classes is isomorphic to a free Bogoliubov crossed product associated with a representation in the other classes.*

- *The class of representations $\lambda \oplus \pi$, where λ is the left regular representation of $\mathbb{Z}$ and π is a faithful almost periodic representation of dimension at least* 2*.*
- *The class of representations $\lambda \oplus \pi$, where λ is the left regular representation of $\mathbb{Z}$ and π is a non-faithful almost periodic representation of dimension at least* 2*.*
- *The class of representations $\rho \oplus \pi$, where ρ is a representation of $\mathbb{Z}$ whose spectral measure μ and all of its convolutions μ^{*n} are non-atomic and singular with respect to the Lebesgue measure on S^1 and π is a faithful almost periodic representation of dimension at least* 2*.*
- *The class of representations $\rho \oplus \pi$, where ρ is a representation of $\mathbb{Z}$ whose spectral measure μ and all of its convolutions μ^{*n} are non-atomic and singular with respect to the Lebesgue measure and π is a non-faithful almost periodic representation of dimension at least* 2*.*
- *Faithful almost periodic representations of dimension at least* 2*.*
- *Non-faithful almost periodic representations of dimension at least* 2*.*
- *The class of representations $\rho \oplus \pi$, where ρ is mixing and* $\dim \pi \leqslant 1$*.*

We also try to characterise strong rigidity of free Bogoliubov crossed products in terms of properties of the representation from which they are constructed. In [112] Houdayer and Shlyakhtenko already proved that any free Bogoliubov crossed product associated with a mixing representation of any discrete group is strongly solid. We are able to amend this result in the case of $\mathbb{Z}$-representations in two directions. Our first result on strong solidity is described in the following theorem.

Theorem (See Theorem 5.E). *Let (π, H) be the direct sum of a mixing representation and a representation of dimension at most one. Then $\Gamma(H, \mathbb{Z}, \pi)''$ is strongly solid.*

If (π, H) is a representation of a discrete group G, we say that a subspace $K \leqslant H$ is rigid for G if there is a sequence $g_n \to \infty$ in G, as $n \to \infty$, such that $\pi(g_n)|_K \to \mathrm{id}_K$ strongly as $n \to \infty$. We make the following observation based on a result by Popa published in [153]. It gives a counterpart to our previous theorem.

Theorem (See Theorem 5.5.4). *Let π be an orthogonal representation of $\mathbb{Z}$ with a rigid subspace of dimension at least two. Then M_π is not solid.*

We conjecture that the previous observation describes the only obstruction to strong solidity.

Conjecture (See Conjecture 5.B). *If (π, H) is an orthogonal representation of $\mathbb{Z}$, then the following are equivalent.*

- $\Gamma(H, \mathbb{Z}, \pi)''$ *is solid.*
- $\Gamma(H, \mathbb{Z}, \pi)''$ *is strongly solid.*
- π *has no rigid subspace of dimension two.*

The results of this work, together with our results presented in Chapter 3, give (partial) classification results for natural classes of von Neumann algebras that are constructed from classical data. It would be interesting to find other natural classes of von Neumann algebras for which one can obtain classification results involving isomorphism and non-isomorphism results at the same time.

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Chapter 2

Isomorphisms and fusion rules of orthogonal free quantum groups and their free complexifications

This chapter is based on [182]. We show that all orthogonal free quantum groups are isomorphic to variants of the free orthogonal Wang algebra, the hyperoctahedral quantum group or the quantum permutation group. We also obtain a description of their free complexification. In particular we complete the calculation of fusion rules of all orthogonal free quantum groups and their free complexifications.

2.1 Introduction

One problem in the theory of compact quantum groups is to find examples whose invariants can be calculated. The fusion rules of a compact quantum group are one of these invariants. Fusion rules give a complete description of equivalence classes of irreducible corepresentations and a decomposition of the tensor product of two of them into irreducible corepresentations. One approach to this problem is given by 'free quantum groups' as defined in [24]. These are orthogonal quantum groups, i.e. subgroups of the free orthogonal Wang algebra,

whose intertwiners can be described by non-crossing partitions.
Given natural numbers k and l the set $\mathrm{Part}(k,l)$ denotes the set of all partitions on two rows with k and l points, respectively. That is, an element $P \in \mathrm{Part}(k,l)$ is a partition of the disjoint union $\{1,...,k\} \sqcup \{1,...,l\}$. Alternatively it can be described by a diagram

$$\left\{ \begin{matrix} \cdot & \cdot & \dots & \cdot \\ & & P & \\ \cdot & \cdot & \dots & \cdot \end{matrix} \right\}$$

connecting the k points in the upper row and the l points in the lower row according to the partition of $\{1,...,k\} \sqcup \{1,...,l\}$. P is called non-crossing if it can be represented by a diagram with no lines crossing. The set of all non-crossing partitions on k and l points is denoted by $\mathrm{NC}(k,l)$.
Let $n,k,l \in \mathbb{N}$ and let (e_i) be the standard basis of $\mathbb{C}^n$. Let $i = (i_1,...,i_k) \in \{1,...,n\}^k$ and $j = (j_1,...,j_l) \in \{1,...,n\}^l$ be multi indices and $P \in \mathrm{Part}(k,l)$. We set $P(i,j) = 1$ if and only if the diagram P joins only equal numbers after writing the entries of i in the upper row of the above diagram and those of j in the lower row. If P connects different numbers set $P(i,j) = 0$.
Using this notation, a partition $P \in \mathrm{Part}(k,l)$ defines a linear map T_P from $(\mathbb{C}^n)^{\otimes k}$ to $(\mathbb{C}^n)^{\otimes l}$ by

$$T_P(e_{i_1} \otimes ... \otimes e_{i_k}) = \sum_{j_1,\dots,j_l} P(i_1,...,i_k; j_1,...,j_l) \cdot e_{j_1} \otimes ... \otimes e_{j_l}.$$

A subspace of $\mathrm{Hom}((\mathbb{C}^n)^{\otimes k}, (\mathbb{C}^n)^{\otimes l})$ is by definition spanned by partitions if it is linearly generated by a family (T_P) where P runs through some subset of $\mathrm{Part}(k,l)$.
In [230] the free unitary Wang algebra

$$A_u(n) := \mathrm{C}^*(u_{ij},\ 1 \leqslant i,j \leqslant n | (u_{ij})_{ij}, (u_{ij}^*)_{ij} \text{ are unitary})$$

and the free orthogonal Wang algebra

$$A_o(n) := \mathrm{C}^*(u_{ij},\ 1 \leqslant i,j \leqslant n | (u_{ij})_{ij} = (u_{ij}^*)_{ij} \text{ is unitary})$$

were introduced. Moreover in [231] the quantum permutation group

$$A_s(n) := \mathrm{C}^* \left(u_{ij},\ 1 \leqslant i,j \leqslant n \,\middle|\, \begin{array}{l} (u_{ij}) = (u_{ij}^*) \text{ is unitary and } u_{ij} \text{ are} \\ \text{partial isometries summing up to one} \\ \text{in every row and every column} \end{array} \right)$$

was defined. Note that “are partial isometries” can be replaced by “are projections”. The three last named algebras are compact matrix quantum groups in the sense of Woronowicz [235].
The following class of quantum groups will be of interest in this chapter.

Definition 2.1.1. Let (A, U) be a compact matrix quantum group. Then it is called free if

- The morphism $(\mathrm{A_u}(n), U_\mathrm{u}) \to (\mathrm{A_s}(n), U_\mathrm{s})$ mapping the entries of U_u to those of U_s factorizes through (A, U).
- The intertwiner spaces $\mathrm{Hom}(U^{i_1} \boxtimes \cdots \boxtimes U^{i_k}, U^{j_1} \boxtimes \cdots \boxtimes U^{j_l})$, $i_\alpha, j_\beta \in \{1, \overline{\ }\}$ are spanned by partitions, where $\overline{U} = (u_{ij}^*)$ is the conjugate corepresentation of U and $\boxtimes$ denotes the tensor product of corepresentations.

If the first condition is strengthened by requiring that the morphism $(\mathrm{A_o}(n), U_\mathrm{o}) \to (\mathrm{A_s}(n), U_\mathrm{s})$ factors through (A, U), then A it is called orthogonal free.

In [24] the following classification was achieved.

Theorem 2.1.2. *There are exactly six orthogonal free quantum groups. Namely*

1. *The free orthogonal Wang algebra.*
2. *The quantum permutation group.*
3. *The hyperoctahedral quantum group*

$$\mathrm{A_h}(n) := C^*\left(u_{ij},\ 1 \leqslant i, j \leqslant n \,\middle|\, \begin{array}{l} (u_{ij}) = (u_{ij}^*) \text{ is unitary and} \\ u_{ij} \text{ are partial isometries} \end{array}\right).$$

4. *The bistochastic quantum group*

$$\mathrm{A_b}(n) := C^*\left(u_{ij},\ 1 \leqslant i, j \leqslant n \,\middle|\, \begin{array}{l} (u_{ij}) = (u_{ij}^*) \text{ is unitary and} \\ u_{ij} \text{ sum up to one} \\ \text{in every row and every column} \end{array}\right).$$

5. *The symmetrized bistochastic quantum group*

$$\mathrm{A_{b'}}(n) := C^*\left(u_{ij},\ 1 \leqslant i, j \leqslant n \,\middle|\, \begin{array}{l} (u_{ij}) = (u_{ij}^*) \text{ is unitary and} \\ u_{ij} \text{ sum up to the same element} \\ \text{in every row and every column} \end{array}\right).$$

6. *The symmetrized quantum permutation group*

$$\mathrm{A_{s'}}(n) := C^*\left(u_{ij},\ 1 \leqslant i, j \leqslant n \,\middle|\, \begin{array}{l} (u_{ij}) = (u_{ij}^*) \text{ is unitary and} \\ u_{ij} \text{ are partial isometries} \\ \text{summing up to the same element} \\ \text{in every row and every column} \end{array}\right).$$

The fusion rules of (1) were calculated in [7], those of (2) in [9] and those of (3) in [25]. We show that the remaining examples are slight modifications of $\mathrm{A_o}(n)$ and $\mathrm{A_s}(n)$. In particular we can derive their fusion rules and find that $\mathrm{A_{b'}}(n)$ and $\mathrm{A_{s'}}(n)$ are counterexamples to a conjecture by Banica and Vergnioux given in [25].
In [12] the free complexification of orthogonal free quantum groups was considered. If (A, U) is a orthogonal free quantum group, then its free complexification $(\widetilde{A}, \widetilde{U})$ is by definition the sub-C*-algebra of the free product $A * \mathcal{C}(S^1)$ generated by the entries of $\widetilde{U} := U \cdot \mathrm{id}_{\mathrm{S}^1} = (u_{ij} \cdot \mathrm{id}_{\mathrm{S}^1})$. Here $\mathrm{id}_{\mathrm{S}^1}$ denotes the canonical generator of $\mathcal{C}(\mathrm{S}^1)$. As Banica shows in [12] the intertwiners between tensor products of the fundamental corepresentation and its conjugate can be described by the intertwiners of the orthogonal free quantum group it comes from. With additional requirements we can calculate the fusion rules of the free complexification from the fusion rules of the original orthogonal free quantum group. These additional requirements are fulfilled by $\mathrm{A_o}(n)$ and $\mathrm{A_h}(n)$, which gives the fusion rules of $\mathrm{A_k}(n) = \widetilde{\mathrm{A_h}(n)}$. Those of $\mathrm{A_u}(n) = \widetilde{\mathrm{A_o}(n)}$ are known from [8].
From [12] we know that $\widetilde{\mathrm{A_b}(n)} = \widetilde{\mathrm{A_{b'}}(n)}$ and $\widetilde{\mathrm{A_s}(n)} = \widetilde{\mathrm{A_{s'}}(n)}$. We denote $\widetilde{\mathrm{A_b}(n)} =: \mathrm{A_c}(n)$ and $\widetilde{\mathrm{A_s}(n)} =: \mathrm{A_p}(n)$. They can be decomposed and described in terms of $\mathrm{A_o}(n)$ and $\mathrm{A_s}(n)$ again.

2.2 Preliminaries

We will mainly work with compact matrix quantum groups as defined by Worono-wicz in [235]. If A is a *-algebra and $U \in \mathrm{M}_n(A)$ we denote by $\overline{U}$ the matrix whose entries are conjugated, i.e. $\overline{U}_{ij} = (U_{ij})^*$.
A pair (A, U) of a C*-algebra A and a unitary $U \in \mathrm{M}_n(A)$ is called a compact matrix quantum group if

- A is generated by the entries of U,
- there is a *-homomorphism $\Delta : A \to A \otimes_{\min} A$ mapping u_{ij} to $\sum_k u_{ik} \otimes u_{kj}$,
- the matrix $\overline{U}$ is invertible.

A morphism of compact matrix quantum groups $(A, U) \xrightarrow{\phi} (B, V)$ is a *-homo-morphism $A \to B$ such that $\phi(u_{ij}) = v_{ij}$ where U and V must have the same size. There is at most one morphism from one quantum group to another. If there is a morphism $(A, U) \to (B, V)$ then we say that (B, V) is a quantum

subgroup of (A, U).
Every compact matrix quantum group is also a compact quantum group, i.e. a C*-algebra A with a *-homomorphism $\Delta : A \to A \otimes_{\min} A$ such that

- $(\Delta \otimes \mathrm{id}) \circ \Delta = (\mathrm{id} \otimes \Delta) \circ \Delta$,
- $\overline{\mathrm{span}}(A \otimes 1)\Delta(A) = \overline{\mathrm{span}}(1 \otimes A)\Delta(A) = A \otimes A$.

A morphism of compact quantum groups $(A, \Delta_A) \xrightarrow{\phi} (B, \Delta_B)$ is a unital *-homomorphism from A to B such that $\Delta_B \circ \phi = (\phi \otimes \phi) \circ \Delta_A$. Every morphism of compact matrix quantum groups is also a morphism of compact quantum groups. We will also refer to a quantum group (A, U) or (A, Δ) as A. If (A, Δ_A) and (B, Δ_B) are quantum groups, then we denote by $(A, \Delta_A) \otimes (B, \Delta_B)$ the direct sum of quantum groups and by $(A, \Delta_A) * (B, \Delta_B)$ their free product. We will also write $A \otimes B$ and $A * B$. A unitary corepresentation matrix of (A, Δ) is a unitary matrix $V \in \mathrm{M}_m(A)$ such that $\Delta(v_{ij}) = \sum_k v_{ik} \otimes v_{kj}$. In particular a one dimensional corepresentation matrix is just a unitary group-like element of A.

2.3 Free fusion rings

In this section we will introduce free fusion rings and prove that they are free unital rings.

We will use the following notation for words in free monoids. Let $M = \mathrm{mon}(S)$ be a free monoid over a set S. If $w \in M$ is a word of length k, then we write w_i for the i-th letter of w, $1 \leqslant i \leqslant k$. Hence $w = w_1 w_2 w_3 \dots w_{k-1} w_k$.

Definition 2.3.1. A free fusion monoid is a free monoid $M = \mathrm{mon}(S)$ over a set S with a fusion $\cdot : S \times S \to S \cup \{\emptyset\}$ and a conjugation $^- : S \longrightarrow S$. They must satisfy the following conditions.

1. The fusion $\cdot$ is associative, where we make the convention that $s \cdot s'$ is the empty set if one of s, s' is the empty set.
2. The conjugation is involutive, i.e. $\overline{\overline{s}} = s$ for all $s \in S$.
3. Fusion and conjugation are compatible in the following sense. For all $s_1, s_2, s_3 \in S$ we have

$$s_1 \cdot s_2 = \overline{s_3} \Leftrightarrow s_2 \cdot s_3 = \overline{s_1}$$

A set S equipped with fusion and conjugation is called a fusion set.
The fusion and conjugation of S induce a fusion and a conjugation on M via

- $w \cdot w' = w_1 \dots w_{k-1}(w_k \cdot w'_1)w'_2 \dots w'_l$ where this fusion is the empty set by convention if $w_k \cdot w'_1 = \emptyset$.
- $\overline{w} = \overline{w_k} \dots \overline{w_1}$

If $M = \text{mon}(S)$ is a free fusion monoid, we can turn $\mathbb{Z}M$ into an associative ring by

$$a_w \cdot a_{w'} = \sum_{\substack{w=xy \\ w'=\overline{y}z}} (a_{xz} + a_{x\cdot z}).$$

Here w, w' are words in M, a_w and $a_{w'}$ are the corresponding elements in $\mathbb{Z}M$, xy, $\overline{y}z$ and xz denote the concatenation of words and the second term in the sum is by convention always ignored if the fusion $x \cdot z$ is empty. Actually condition (3) of the previous definition is a necessary condition for making $\mathbb{Z}M$ associative, as it can be seen by considering $(a_{s_1} \cdot a_{s_2}) \cdot a_{s_3} = a_{s_1} \cdot (a_{s_2} \cdot a_{s_3})$ for $s_1, s_2, s_3 \in S$. A *-ring isomorphic to $\mathbb{Z}M$ for some fusion monoid M is called a free fusion ring.
From the point of view of rings, free fusion rings are very easy. Actually they are free. The proof of the following lemma was already given in [25] in some special cases.

Lemma 2.3.2. *A free fusion ring over a fusion set S is the free unital ring over a_s, $s \in S$.*

Proof. Let $\mathbb{Z}M$ be the fusion ring over a fusion set S. It suffices to show that $\mathbb{Z}M$ is a free $\mathbb{Z}$-module with the basis $a_{s_1} \cdots a_{s_k}$ with $k \in \mathbb{N}$ and $s_1, \dots, s_k \in S$. So it suffices to express the elements of the $\mathbb{Z}$-basis $a_w, w \in M$ as $\mathbb{Z}$-linear combinations of the elements $a_{s_1} \cdots a_{s_k}$ with $k \in \mathbb{N}$ and $s_1, \dots, s_k \in S$ and to show that $\{a_{s_1} \cdots a_{s_k} | k \in \mathbb{N}, s_1, \dots, s_k \in S\}$ is $\mathbb{Z}$-linearly independent.
There are coefficients $C^w_{s_1 \dots s_k} \in \mathbb{Z}$ such that $a_{s_1} \cdots a_{s_k} = a_{s_1 \dots s_k} + \sum_{|w|<k} C^w_{s_1 \dots s_k} a_w$, where $|w|$ is the length of the word $w \in M$. This shows that $\{a_{s_1} \cdots a_{s_k} | k \in \mathbb{N}, s_1, \dots, s_k \in S\}$ is linearly independent. Moreover, by induction on k there are coefficients $D^w_{s_1 \dots s_l} \in \mathbb{Z}$ such that $a_{s_1 \dots s_k} = a_{s_1} \cdots a_{s_k} + \sum_{|w|<k} D^w_{s_1 \dots s_k} a_{w_1} \cdots a_{w_{|w|}}$. This shows that all $a_w, w \in M$ are linear combinations of $a_{s_1} \cdots a_{s_k}$ with $k \in \mathbb{N}$ and $s_1, \dots, s_k \in S$. □

Remark 2.3.3. Free fusion rings can be used to describe fusion rules very shortly and there is hope to use free fusion rings as a starting point for proofs of several properties of quantum groups. See Section 10 of [25] for a comment on these possibilities. However in order to justify the concept of free fusion rings intrinsically it would be good to answer the following question affirmatively. Is every fusion ring of a compact quantum group that is free as a unital ring a free fusion ring?

2.4 Some isomorphisms of combinatorial quantum groups

In this section we will consider combinatorial quantum groups $\mathrm{A}_*(n)$ for $* \in \{b, b', s', c, p\}$. They are free products or direct sums of known quantum groups. For $* \in \{b', s', c, p\}$ it turns out that their fusion rings are not free.

Theorem 2.4.1. *We have the following isomorphisms of compact quantum groups (not necessarily preserving the fundamental corepresentation).*

1. $\mathrm{A_b}(n)$ *is isomorphic to* $\mathrm{A_o}(n-1)$.
2. $\mathrm{A_{s'}}(n)$ *is isomorphic to the direct sum* $\mathrm{A_s}(n) \otimes C^*(\mathbb{Z}/2\mathbb{Z})$.
3. $\mathrm{A_{b'}}(n)$ *is isomorphic to the free product* $\mathrm{A_b}(n) * C^*(\mathbb{Z}/2\mathbb{Z})$.
4. $\mathrm{A_p}(n)$ *is isomorphic to the free product* $\mathrm{A_s}(n) * \mathcal{C}(\mathrm{S}^1)$.
5. $\mathrm{A_c}(n)$ *is isomorphic to the free product* $\mathrm{A_b}(n) * \mathcal{C}(\mathrm{S}^1)$.

Remark 2.4.2. Note that in the case $n \leqslant 3$ we have the isomorphisms $\mathrm{A_s}(n) \cong \mathcal{C}(\mathrm{S_n})$ and $\mathrm{A_o}(1) \cong \mathcal{C}(\{-1, 1\})$. So the given descriptions can be further simplified.

Theorem 2.4.1(1) is proven by the following remark. Let $U \in \mathrm{M}_n(A)$ be an orthogonal matrix, i.e. $\overline{U} = U$ unitary, where A is any unital C*-algebra. Then U is bistochastic if and only if the vector $(1, 1, \ldots, 1)^{\mathrm{t}}$ is a right eigenvector and $(1, 1, \ldots, 1)$ is a left eigenvector of U. If $T \in \mathrm{M}_n(\mathbb{C})$ denotes any orthogonal matrix such that $T(1, 0, \ldots, 0)^{\mathrm{t}} = (1/\sqrt{n}, \ldots, 1/\sqrt{n})^{\mathrm{t}}$, then an orthogonal matrix U is bistochastic if and only if $T^{\mathrm{t}}UT$ is of block form with 1 in the upper left corner and an orthogonal $(n-1) \times (n-1)$ matrix in the lower right corner. The key observation for the rest of 2.4.1 is the following lemma.

Lemma 2.4.3. *Let* $* \in \{b', s', c, p\}$. *The fundamental corepresentation of* $\mathrm{A}_*(n)$ *contains a one dimensional non-trivial corepresentation* U_z *which fulfils* $U_z \boxtimes \overline{U_z} \simeq 1$. *If* $* \in \{b', s'\}$ *then* $U_z \simeq \overline{U_z}$.

Proof. Consider $* = b', s'$ first. The element $z = \sum_i u_{ij}$ is easily seen to be a unitary group-like element, so it corresponds to a one dimensional unitary

corepresentation of $\mathrm{A}_*(n)$. Consider the group $\mathrm{S}_n \oplus \mathbb{Z}/2\mathbb{Z} \subset U_n$ as permutation matrices with entries $+1$ and -1. Let $U_{\mathrm{S}_n \oplus \mathbb{Z}/2\mathbb{Z}}$ be the canonical fundamental corepresentation of $\mathcal{C}(\mathrm{S}_n \oplus \mathbb{Z}/2\mathbb{Z})$. Then the image of z under the map $(\mathrm{A}_*(n), U_*) \to (\mathcal{C}(\mathrm{S}_n \oplus \mathbb{Z}/2\mathbb{Z}), U_{\mathrm{S}_n \oplus \mathbb{Z}/2\mathbb{Z}})$ is -1, so z is non-trivial.
For $* = p, c$ consider $z := \mathrm{id}_{\mathrm{S}^1}$ as coming from the copy of $\mathcal{C}(\mathrm{S}^1)$. This copy is contained in $\mathrm{A}_*(n)$, since the trivial corepresentation is contained in the fundamental corepresentation of $\mathrm{A}_\mathrm{b}(n)$ and $\mathrm{A}_\mathrm{s}(n)$.
Using the relations of $\mathrm{A}_*(n)$ we can check the rest of the claim by simple calculations. □

Remark 2.4.4. The last lemma shows, that the fusion rules of neither of the quantum groups $\mathrm{A}_*(n)$ for $\in \{b', s', c, p\}$ can be described by a free fusion ring. Actually in a free fusion ring any element $a \neq 1$ satisfies $a \cdot a^* \neq 1$. This gives two counterexamples to the conjecture that for $n \geqslant 4$ the fusion rules of all orthogonal free quantum groups can be described by a free fusion ring, which was stated in [25].

Remark 2.4.5. The fundamental corepresentation of any matrix quantum group that has $(\mathrm{A}_\mathrm{s}(n), U_\mathrm{s})$ as a sub quantum group cannot be the sum of more than two irreducible corepresentations. In particular the last lemma already gives a decomposition $U \simeq U_z \boxplus V$ with U_z non-trivial and one dimensional and V irreducible, where U is the fundamental corepresentation of $\mathrm{A}_*(n)$.

Proof of Theorem 2.4.1. The isomorphism of (2) is given by $\mathrm{A}_\mathrm{s}(n) \otimes C^*(\mathbb{Z}/2\mathbb{Z}) \to \mathrm{A}_{\mathrm{s'}}(n) : u^{\mathrm{s}}_{ij} \otimes 1 \mapsto u^{\mathrm{s}'}_{ij} \cdot z,\ 1 \otimes u_{\overline{1}} \mapsto z$. This map exists since z is central in $\mathrm{A}_{\mathrm{s'}}(n)$ as an easy calculation shows. The inverse map is given by

$$\mathrm{A}_{\mathrm{s'}}(n) \to \mathrm{A}_\mathrm{s}(n) \otimes C^*(\mathbb{Z}/2\mathbb{Z}) : u^{\mathrm{s'}}_{ij} \to u^{\mathrm{s}}_{ij} \otimes u_{\overline{1}}.$$

In order to prove (3) we use again an orthogonal matrix $T \in \mathrm{M}_n(\mathbb{C})$ such that $T(1, 0, ..., 0)^\mathrm{t} = (1/\sqrt{n}, ..., 1/\sqrt{n})^\mathrm{t}$. Then a matrix $U \in \mathrm{M}_n(A)$ for some C*-algebra A satisfies the relations of $U_{\mathrm{b'}}$ if and only if $T^\mathrm{t} U T$ is a block matrix with a self-adjoint unitary in the upper left corner and an orthogonal $(n-1) \times (n-1)$ matrix in the lower right corner. This proves $\mathrm{A}_{\mathrm{b'}}(n) \cong \mathrm{A}_\mathrm{o}(n-1) * C^*(\mathbb{Z}/2\mathbb{Z}) \cong \mathrm{A}_\mathrm{b}(n) * C^*(\mathbb{Z}/2\mathbb{Z})$.
The isomorphism of (4) is given by

$$\mathrm{A}_\mathrm{s}(n) * \mathcal{C}(\mathrm{S}^1) \to \mathrm{A}_\mathrm{p}(n) : u^{\mathrm{s}}_{ij} \mapsto u^{\mathrm{p}}_{ij} \cdot z^*,\ \mathrm{id}_{\mathrm{S}^1} \mapsto z.$$

The isomorphism of (5) is given by

$$\mathrm{A}_\mathrm{b}(n) * \mathcal{C}(\mathrm{S}^1) \to \mathrm{A}_\mathrm{c}(n) : u^{\mathrm{b}}_{ij} \mapsto u^{\mathrm{c}}_{ij} \cdot z^*,\ \mathrm{id}_{\mathrm{S}^1} \mapsto z.$$

All the isomorphisms respect the comultiplication, since z is group-like. Hence, they are isomorphisms of quantum groups. □

2.5 Fusion rules for free products and the quantum group $\mathbf{A_k}(n)$

In this section we describe the fusion rules of the free complexification $\mathrm{A_k}(n) \cong \widetilde{\mathrm{A_h}(n)}$. Instead of referring to $\mathrm{A_k}(n)$ explicitly, we will work in a more general setting and deduce its fusion rules as a corollary. Roughly the main statement of this section is given by the following theorem. See theorem 2.5.5 for a precise statement.

Theorem 2.5.1. *Let (A, U) be an orthogonal compact matrix quantum group, i.e. $\overline{U} = U$, such that its fusion rules are free. Assume further that $1 \notin U^{\boxtimes 2k+1}$ for any $k \in \mathbb{N}$. Then the fusion rules of $(\widetilde{A}, \widetilde{U})$ are free and can be described in terms of the fusion rules of (A, U).*

The following theorem is due to Wang [231].

Theorem 2.5.2. *Let (A, Δ_A) and (B, Δ_B) be compact quantum groups. Let $(U^\alpha)_{\alpha \in \mathscr{A}}$ and $(U^\beta)_{\beta \in \mathscr{B}}$ be complete sets of representatives of irreducible corepresentations of A and B, respectively. Then the corepresentations $(W^{\gamma_1} \boxtimes \cdots \boxtimes W^{\gamma_n})$ with $n \in \mathbb{N}$, all W^{γ_i} in $\{U^\alpha \mid \alpha \in \mathscr{A}\}$ and $\{U^\beta \mid \beta \in \mathscr{B}\}$ and neighbours not from the same set, form a complete set of irreducible representations of the free product $(A, \Delta_A) * (B, \Delta_B)$.*

The following observation will be useful when studying the fusion rules of a free complexification.

Remark 2.5.3. Let $A * B$ be a free product of compact quantum groups with irreducible corepresentations $W^{\gamma_1} \boxtimes \cdots \boxtimes W^{\gamma_n}$ and $W^{\delta_1} \boxtimes \cdots \boxtimes W^{\delta_m}$ as in the last theorem. Then

1. If W^{γ_n} and W^{δ_1} are not corepresentations of the same factor of the free product, then $W^{\gamma_1} \boxtimes \cdots \boxtimes W^{\gamma_n} \boxtimes W^{\delta_1} \boxtimes \cdots \boxtimes W^{\delta_m}$ is an irreducible corepresentation of $A * B$.

2. If W^{γ_n} and W^{δ_1} are corepresentations of the same factor and $W^{\gamma_n} \boxtimes W^{\delta_1} = \sum_{i=1}^{k} W^{\epsilon_i} + \delta_{\overline{W^{\gamma_n}}, W^{\delta_1}} \cdot 1$ is the decomposition into irreducible corepresen-

tations, then

$$\begin{aligned} &W^{\gamma_1} \boxtimes \cdots \boxtimes W^{\gamma_n} \boxtimes W^{\delta_1} \boxtimes \cdots \boxtimes W^{\delta_m} \\ &\quad = \sum_{i=1}^{k} (W^{\gamma_1} \boxtimes \cdots \boxtimes W^{\gamma_{n-1}} \boxtimes W^{\epsilon_i} \boxtimes W^{\delta_2} \boxtimes \cdots \boxtimes W^{\delta_m}) \\ &\qquad + \delta_{\overline{W^{\gamma_n}}, W^{\delta_1}} \cdot W^{\gamma_1} \boxtimes \cdots \boxtimes W^{\gamma_{n-1}} \boxtimes W^{\delta_2} \boxtimes \cdots \boxtimes W^{\delta_m} \end{aligned}$$

and the first k summands of this decomposition are irreducible.

For the rest of this section fix an orthogonal compact matrix quantum group (A, U) such that its fusion rules are described by a free fusion ring over the fusion set S. Assume further that $1 \notin U^{\boxtimes 2k+1}$ for any $k \in \mathbb{N}$.
Note that the fusion ring of $\widetilde{A}$ is the fusion subring of $\mathrm{Rep}(A * \mathcal{C}(\mathrm{S}^1))$ that is generated by $U \boxtimes z$, where z denotes the identity on the circle.
We will construct the free complexification $\widetilde{S}$ of S and prove that the fusion rules of $(\widetilde{A}, \widetilde{U})$ are described by $\widetilde{S}$. We begin by constructing $\widetilde{S}$.
Let $\mathrm{Rep}^{\mathrm{irr}}_{\mathrm{even}}$ (respectively $\mathrm{Rep}^{\mathrm{irr}}_{\mathrm{odd}}$) be the set of classes of irreducible corepresentations of A that appear as subrepresentations of an even (respectively odd) tensor power of U. We have $\mathrm{Rep}^{\mathrm{irr}}_{\mathrm{even}} \cap \mathrm{Rep}^{\mathrm{irr}}_{\mathrm{odd}} = \emptyset$ due to Frobenius duality and the requirement $1 \notin U^{2k+1}$ for all $k \in \mathbb{N}$. Let $S_{\mathrm{even}} \subset S$ (resp. $S_{\mathrm{odd}} \subset S$) be the set of elements corresponding to corepresentations from $\mathrm{Rep}^{\mathrm{irr}}_{\mathrm{even}}$ (resp. $\mathrm{Rep}^{\mathrm{irr}}_{\mathrm{odd}}$). The set $\widetilde{S}$ is then by definition the disjoint union $S_{\mathrm{even}} \sqcup S_{\mathrm{even}} \sqcup S_{\mathrm{odd}} \sqcup S_{\mathrm{odd}}$. Denote the first copy of S_{even} (resp. S_{odd}) by $S^{(1)}_{\mathrm{even}}$ (resp. $S^{(1)}_{\mathrm{odd}}$) and the second one by $S^{(2)}_{\mathrm{even}}$ (resp. $S^{(2)}_{\mathrm{odd}}$).

What follows is motivated by the following point of view:

Remark 2.5.4. We consider element of $S^{(1)}_{\mathrm{even}}$ as a plain copy of those in S_{even}. The elements of $S^{(2)}_{\mathrm{even}}$ are of the form $z^* \cdot s \cdot z$ for some $s \in S_{\mathrm{even}}$. Similarly we consider elements of $S^{(1)}_{\mathrm{odd}}$ as $s \cdot z$ and elements of $S^{(2)}_{\mathrm{odd}}$ as $z^* \cdot s$ for $s \in S_{\mathrm{odd}}$.

Define a conjugation on $\widetilde{S}$ by the conjugation on S leaving $S^{(1)}_{\mathrm{even}}$ and $S^{(2)}_{\mathrm{even}}$ globally invariant and exchanging $S^{(1)}_{\mathrm{odd}}$ and $S^{(2)}_{\mathrm{odd}}$. Note that $\overline{S_{\mathrm{even}}} = S_{\mathrm{even}}$ and $\overline{S_{\mathrm{odd}}} = S_{\mathrm{odd}}$, i.e. the conjugation on $\widetilde{S}$ is well defined. A fusion on $\widetilde{S}$ can be defined according to the following table.

$\cdot$	$S^{(1)}_{\text{even}}$	$S^{(2)}_{\text{even}}$	$S^{(1)}_{\text{odd}}$	$S^{(2)}_{\text{odd}}$
$S^{(1)}_{\text{even}}$	$S^{(1)}_{\text{even}} \cup \{\varnothing\}$	$\varnothing$	$S^{(1)}_{\text{odd}} \cup \{\varnothing\}$	$\varnothing$
$S^{(2)}_{\text{even}}$	$\varnothing$	$S^{(2)}_{\text{even}} \cup \{\varnothing\}$	$\varnothing$	$S^{(2)}_{\text{odd}} \cup \{\varnothing\}$
$S^{(1)}_{\text{odd}}$	$\varnothing$	$S^{(2)}_{\text{odd}} \cup \{\varnothing\}$	$\varnothing$	$S^{(1)}_{\text{even}} \cup \{\varnothing\}$
$S^{(2)}_{\text{odd}}$	$S^{(2)}_{\text{odd}} \cup \{\varnothing\}$	$\varnothing$	$S^{(2)}_{\text{even}} \cup \{\varnothing\}$	$\varnothing$

The row gives the element which is fused from the right with an element coming from the set indicated by the column. The fusion is empty if this is indicated by the table and is otherwise the usual fusion of two elements of S lying in the part of $\tilde{S}$ indicated by the table. Note that this definition makes sense, since $S_{\text{even}} \cdot S_{\text{even}}, S_{\text{odd}} \cdot S_{\text{odd}} \subset S_{\text{even}} \cup \{\varnothing\}$ and $S_{\text{even}} \cdot S_{\text{odd}}, S_{\text{odd}} \cdot S_{\text{even}} \subset S_{\text{odd}} \cup \{\varnothing\}$. It is easy to see that $\tilde{S}$ with this structure is a fusion set.
Now we can state a precise version of 2.5.1.

Theorem 2.5.5. *Let (A, U) be an orthogonal compact matrix quantum group such that its fusion rules are described by a free fusion ring over the fusion set S. Assume further that $1 \notin U^{\boxtimes 2k+1}$ for any $k \in \mathbb{N}$. Then the fusion rules of $(\tilde{A}, \tilde{U})$ are given by the free complexification $\tilde{S}$ of S.*

We construct a complete set of corepresentations of $\tilde{A}$. In order to do so we associate an irreducible corepresentations of $(\tilde{A}, \tilde{U})$ to any element of $\tilde{R} := \text{Rep}^{\text{irr}}_{\text{even}} \sqcup \text{Rep}^{\text{irr}}_{\text{even}} \sqcup \text{Rep}^{\text{irr}}_{\text{odd}} \sqcup \text{Rep}^{\text{irr}}_{\text{odd}}$. We denote the i-th copy of $\text{Rep}^{\text{irr}}_{\text{even}}$ ($\text{Rep}^{\text{irr}}_{\text{odd}}$) by $\text{Rep}^{\text{irr},(i)}_{\text{even}}$ ($\text{Rep}^{\text{irr},(i)}_{\text{odd}}$). Let V be a irreducible corepresentation in $\text{Rep}^{\text{irr}}_{\text{even}}$. Then V and $z^* \cdot V \cdot z$ are corepresentations of $\tilde{A}$. Actually, if V is an irreducible subrepresentation of $U^{\boxtimes 2k}$ then V is an irreducible subrepresentation of $(\tilde{U} \boxtimes \overline{\tilde{U}})^{\boxtimes k}$ and $z^* \cdot V \cdot z$ is an irreducible subrepresentation of $(\overline{\tilde{U}} \boxtimes \tilde{U})^{\boxtimes k}$. We consider V as an element of $\text{Rep}^{\text{irr},(1)}_{\text{even}}$ and $z^* \cdot V \cdot z$ as an element of $\text{Rep}^{\text{irr},(2)}_{\text{even}}$. Similarly we see that if $V \in \text{Rep}^{\text{irr}}_{\text{odd}}$ then we can associate with it corepresentations $V \cdot z \in \text{Rep}^{\text{irr},(1)}_{\text{odd}}$ and $z^* \cdot V \in \text{Rep}^{\text{irr},(2)}_{\text{odd}}$. Note that elements s from $\tilde{S}$ give corepresentations $\tilde{U}_s$ by this identification. Consider a word $w = w_1 \dots w_k$ with letters in $\tilde{R}$. We say that w is reduced if in the sequence $\tilde{U}_{w_1}, \dots, \tilde{U}_{w_n}$ a z is never followed by z^* and U_x is always followed by z or z^*. In formal terms:

$$\forall 1 \leqslant i \leqslant k-1 : (w_i \in \text{Rep}^{\text{irr},(1)}_{\text{even}} \cup \text{Rep}^{\text{irr},(2)}_{\text{odd}} \Rightarrow w_{i+1} \in \text{Rep}^{\text{irr},(2)}_{\text{even}} \cup \text{Rep}^{\text{irr},(2)}_{\text{odd}}) \wedge$$
$$(w_i \in \text{Rep}^{\text{irr},(2)}_{\text{even}} \cup \text{Rep}^{\text{irr},(1)}_{\text{odd}} \Rightarrow w_{i+1} \in \text{Rep}^{\text{irr},(1)}_{\text{even}} \cup \text{Rep}^{\text{irr},(1)}_{\text{odd}})$$

Any such reduced word $w = w_1 \dots w_k$ gives rise to an irreducible corepresentation of $\tilde{A}$ by $\tilde{U}_w := \tilde{U}_{w_1} \boxtimes \dots \boxtimes \tilde{U}_{w_k}$ and different reduced words give rise to inequivalent corepresentations by 2.5.2. Since any iterated tensor product of

$\widetilde{U}$ and $\overline{\widetilde{U}}$ decomposes as a sum of irreducible corepresentations of the type $\widetilde{U}_w$, where w is a reduced word with letters in $\widetilde{R}$, any irreducible corepresentation of $\widetilde{A}$ is equivalent to some $\widetilde{U}_w$.

Definition 2.5.6. Consider now a word $w = w_1 \dots w_k$ with letters in $\widetilde{S}$. It is called connected if every z is followed by a z^*. Formally:

$$\forall 1 \leqslant i \leqslant k-1 : (w_i \in S^{(1)}_{\text{even}} \cup S^{(2)}_{\text{odd}} \Rightarrow w_{i+1} \in S^{(1)}_{\text{even}} \cup S^{(1)}_{\text{odd}}) \wedge$$
$$(w_i \in S^{(2)}_{\text{even}} \cup S^{(1)}_{\text{odd}} \Rightarrow w_{i+1} \in S^{(2)}_{\text{even}} \cup S^{(2)}_{\text{odd}})$$

The following definition says how we can associate irreducible corepresentations of $\widetilde{A}$ to words with letters in $\widetilde{S}$.

Definition 2.5.7. If w is an arbitrary word with letters in $\widetilde{S}$ then it has a unique decomposition $w = x_1 \dots x_l$ into maximal connected words. This gives rise to a unique reduced word w' with letters in $\widetilde{R}$. We set $\widetilde{U}_w := \widetilde{U}_{w'}$

Next we have to do some preparations in order to prove Theorem 2.5.5.

Definition 2.5.8. Let $x = x_1 \dots x_m$ be a word in $\widetilde{S}$. Then $\check{x_i}$ is the letter in S corresponding to x_i and $\check{x} := \check{x_1}\check{x_2}\dots\check{x_m}$.

Remark 2.5.9. Note that if x is a connected word with letters in S then according to remark 2.5.4 it can be written as $z^{i_0} \cdot \check{x} \cdot z^{i_1}$, $i_0, i_1 \in \{0, 1, -1\}$ and we have $\widetilde{U}_x = z^{i_0} \boxtimes U_{\check{x}} \boxtimes z^{i_1}$.

Definition 2.5.10. Let x, y be connected words with letters in $\widetilde{S}$. We say that (x, y) fits together if xy is a connected word.

Lemma 2.5.11. *Let $x = x_1 \dots x_m$ and $y = y_1 \dots y_n$ be connected words with letters in $\widetilde{S}$ such that (x_m, y_1) fits together. Write $\widetilde{U}_x = z^{i_0} \boxtimes U_{\check{x}} \boxtimes z^{i_1}$ and $\widetilde{U}_y = z^{j_0} \boxtimes U_{\check{y}} \boxtimes z^{j_1}$. Then*

$$\widetilde{U}_x \boxtimes \widetilde{U}_y = z^{i_0} \boxtimes \left(\sum_{x=ac, y=\overline{c}b} U_{\check{a}\check{b}} \boxplus U_{\check{a}\cdot\check{b}} \right) \boxtimes z^{j_1} = \sum_{x=ac, y=\overline{c}b} \widetilde{U}_{ab} \boxplus \widetilde{U}_{a\cdot b}.$$

Proof. Since (x, y) fits together, we have $z^{i_1} \boxtimes z^{j_0} = 1$. So by Remark 2.5.3 the first equation follows. We have to prove that for all $x = ac$, $y = \overline{c}b$

1. $z^{i_0} \boxtimes U_{\check{a}\check{b}} \boxtimes z^{j_1} = \widetilde{U}_{ab}$
2. $z^{i_0} \boxtimes U_{\check{a}\cdot\check{b}} \boxtimes z^{j_1} = \widetilde{U}_{a\cdot b}$.

In order to prove (1), note that ab is connected, since a, b are connected and (a, b) fits together. So (1) follows from the way irreducible corepresentations are associated to connected words remarked in 2.5.9.
For (2) note that, since (a, b) fits together, $\check{a} \cdot \check{b} = \emptyset$ if and only if $a \cdot b = \emptyset$. If $a \cdot b \neq \emptyset$ then it is connected and (2) follows by remark 2.5.9 again. □

Now we can give the proof of Theorem 2.5.5

Proof of Theorem 2.5.5. Let $x = x_1 \dots x_k$ and $y = y_1 \dots y_l$ be words with letters in $\widetilde{S}$. We have to show that

$$\widetilde{U}_x \boxtimes \widetilde{U}_y = \sum_{x=ac, y=\bar{c}b} \widetilde{U}_{ab} \boxplus \widetilde{U}_{a \cdot b}$$

Let $x = u_1 \dots u_m$ and $y = v_1 \dots v_n$ be the decomposition in maximal connected words. We identify them with letters in $\widetilde{R}$. Then

$$\widetilde{U}_x = z^{i_0} \boxtimes U_{\check{u_1}} \boxtimes z^{i_1} \boxtimes U_{\check{u_2}} \boxtimes z^{i_2} \boxtimes \cdots \boxtimes U_{\check{u_{m-1}}} \boxtimes z^{i_{m-1}} \boxtimes \underbrace{z^{i_m} \boxtimes U_{\check{u_m}} \boxtimes z^{i_{m+1}}}_{=\widetilde{U}_{u_m}},$$

$$\widetilde{U}_y = \underbrace{z^{j_0} \boxtimes U_{\check{v_1}} \boxtimes z^{j_1}}_{=\widetilde{U}_{v_1}} \boxtimes z^{j_2} \boxtimes U_{\check{v_2}} \boxtimes \cdots \boxtimes z^{j_{n-1}} \boxtimes U_{\check{v_{n-1}}} \boxtimes z^{j_n} \boxtimes U_{\check{v_n}} \boxtimes z^{j_{n+1}}$$

with $i_1, ..., i_{m-2}, j_3, ..., j_n \in \{1, *\}$, $i_0, i_m, j_0, j_2 \in \{0, *\}$ and $i_{m-1}, i_{m+1}, j_1, j_{n+1} \in \{0, 1\}$.
We are going to consider the two cases (x_k, y_1) do or do not fit together. Assume that (x_k, y_1) do not fit together. This means $z^{i_{m+1}} \cdot z^{j_0} \neq 1$. Then $\widetilde{U}_x \boxtimes \widetilde{U}_y$ is irreducible by Theorem 2.5.2. Moreover, $xy = u_1 \dots u_m v_1 \dots v_n$ is a decomposition in maximal connected words. So $\widetilde{U}_x \boxtimes \widetilde{U}_y = \widetilde{U}_{xy}$. On the other hand (x_k, y_1) not fitting together implies $x_k \neq \overline{y_1}$ and $x_k \cdot y_1 = \emptyset$. So $\sum_{x=ac, y=\bar{c}b} \widetilde{U}_{ab} \boxplus \widetilde{U}_{a \cdot b} = \widetilde{U}_{xy}$. This completes the proof for the first case.
Assume now that (x_k, y_1) fits together. This means $z^{i_{m+1}} \cdot z^{j_0} = 1$. By Lemma

2.5.11

$$\widetilde{U}_x \boxtimes \widetilde{U}_y = z^{i_0} \boxtimes U_{\check{u_1}} \boxtimes z^{i_1} \boxtimes \cdots \boxtimes U_{\check{u_{m-1}}} \boxtimes z^{i_{m-1}} \boxtimes$$

$$\Big(\sum_{u_m = ac, v_1 = \bar{c}b} \widetilde{U}_{ab} \boxplus \widetilde{U}_{a \cdot b} \Big) \boxtimes z^{j_2} \boxtimes U_{\check{v_2}} \boxtimes z^{j_3} \boxtimes \cdots \boxtimes U_{\check{v_n}} \boxtimes z^{j_{n+1}}$$

$$= z^{i_0} \boxtimes U_{\check{u_1}} \boxtimes z^{i_1} \boxtimes \cdots \boxtimes U_{\check{u_{m-1}}} \boxtimes z^{i_{m-1}} \boxtimes$$

$$\Big(\Big(\sum_{u_m = ac, v_1 = \bar{c}b, |a| \geqslant 1 \text{ or } |b| \geqslant 1} \widetilde{U}_{ab} \boxplus \widetilde{U}_{a \cdot b} \Big) \boxplus \delta_{u_m, \overline{v_1}} \cdot 1 \Big) \boxtimes$$

$$z^{j_2} \boxtimes U_{\check{v_2}} \boxtimes z^{j_3} \boxtimes \cdots \boxtimes U_{\check{v_n}} \boxtimes z^{j_{n+1}}.$$

By applying the induction hypothesis to the term

$$z^{i_0} \boxtimes U_{\check{u_1}} \boxtimes \cdots \boxtimes z^{i_{m-1}} \boxtimes \delta_{u_m, \overline{v_1}} \cdot 1 \boxtimes z^{j_2} \boxtimes U_{\check{v_2}} \boxtimes \cdots \boxtimes z^{j_{n+1}}$$

$$= \delta_{u_m, \overline{v_1}} \cdot \widetilde{U}_{u_1 u_2 \ldots u_{m-1}} \boxtimes \widetilde{U}_{v_2 v_3 \ldots v_n}$$

we obtain

$$\widetilde{U}_x \boxtimes \widetilde{U}_y = \sum_{x = ac, y = \bar{c}b} \widetilde{U}_{ab} \boxplus \widetilde{U}_{a \cdot b}.$$

□

We are now going to deduce the fusion rules of $\mathrm{A_k}(n)$. The following result is proven in [25] and describes the fusion rules of $\mathrm{A_h}(n)$.

Theorem 2.5.12. *Let $S_\mathrm{h} := \{u, p\}$ with fusion $u \cdot u = p \cdot p = p$, $u \cdot p = p \cdot u = u$ and trivial conjugation. The fusion rules of $(\mathrm{A_h}(n), U_\mathrm{h})$ are given by the free fusion ring over S_h in such a way that $U_u \simeq U_\mathrm{h}$ and $U_p \boxplus 1 \simeq (u_{ij}^2)$.*

Using this theorem we obtain the following corollary in the case $A = \mathrm{A_k}(n)$.

Corollary 2.5.13. *The irreducible corepresentations of $\mathrm{A_k}(n)$ are described by the fusion set $S_\mathrm{k} := \{u, v, p, q\}$ with fusion given by*

$\cdot$	u	v	p	q
u	$\varnothing$	q	u	$\varnothing$
v	p	$\varnothing$	$\varnothing$	v
p	$\varnothing$	v	p	$\varnothing$
q	u	$\varnothing$	$\varnothing$	q

and conjugation $\overline{u} = v$, $\overline{p} = p$, $\overline{q} = q$.
The elements of S_k *correspond to the following corepresentations.*

- *The class of the fundamental corepresentation* U *is* U_u.
- *The class of* $\overline{U}$ *is* U_v.
- *The class of the corepresentation* $(u_{ij}^* \cdot u_{ij})$ *is* $U_p \boxplus 1$
- *The class of the corepresentation* $(u_{ij} \cdot u_{ij}^*)$ *is* $U_q \boxplus 1$

Proof. We only have to prove the part about the concrete description of U_u, U_v, U_p and U_q. The fact that U_u is the class of the fundamental corepresentation is obvious from the construction. $U_v \simeq \overline{U}$ follows directly.
It is easy to check that $(u_{ij}^* \cdot u_{ij})$ and $(u_{ij} \cdot u_{ij}^*)$ are corepresentation of $\mathrm{A_k}(n)$. We have the decomposition $U \boxtimes \overline{U} \simeq U_{uv} \boxplus U_p \boxplus 1$. Moreover the construction in this section shows that U_{uv} is $n^2 - n$ dimensional and U_p is $n-1$ dimensional. Since $(u_{ij} \cdot u_{ij}^*)$ is non trivial, it suffices to give at least two linearly independent intertwiners from the n dimensional corepresentation $(u_{ij} \cdot u_{ij}^*)$ to $U \boxtimes \overline{U}$. Two such intertwiners are $\mathbb{C}^n \to (\mathbb{C}^n)^{\otimes 2} : e_i \mapsto e_i \otimes e_i$ and $\mathbb{C}^n \to (\mathbb{C}^n)^{\otimes 2} : e_i \mapsto \sum_j e_j \otimes e_j$.
The proof for $(u_{ij} \cdot u_{ij}^*)$ works similarly. □

Chapter 3

Stable orbit equivalence of Bernoulli actions of free groups and isomorphism of some of their factor actions

This chapter is based on our joint work with Niels Meesschaert and Stefaan Vaes [138]. We give an elementary proof for Lewis Bowen's theorem saying that two Bernoulli actions of two free groups, each having arbitrary base probability spaces, are stably orbit equivalent. Our methods also show that for all compact groups K and every free product Γ of infinite amenable groups, the factor $\Gamma \curvearrowright K^\Gamma/K$ of the Bernoulli action $\Gamma \curvearrowright K^\Gamma$ by the diagonal K-action, is isomorphic with a Bernoulli action of Γ.

3.1 Introduction

Free, ergodic and probability measure preserving (p.m.p.) actions $\Gamma \curvearrowright (X, \mu)$ of countable groups give rise to II_1 factors $\mathrm{L}^\infty(X) \rtimes \Gamma$ through the group measure space construction of Murray and von Neumann. It was shown in [196] that the isomorphism class of the II_1 factor $\mathrm{L}^\infty(X) \rtimes \Gamma$ only depends on the orbit equivalence relation on (X, μ) given by $\Gamma \curvearrowright (X, \mu)$. This led Dye in [69] to a systematic study of group actions up to orbit equivalence, where he proved the fundamental result that all free ergodic p.m.p. actions of $\mathbb{Z}$ are orbit equivalent. Note that two such actions need not be isomorphic (using entropy, spectral measure, etc). In [151] Ornstein and Weiss showed that actually all orbit equivalence relations of all free ergodic p.m.p. actions of infinite amenable groups are isomorphic with the unique ergodic hyperfinite equivalence relation of type II_1.

The nonamenable case is far more complex and many striking rigidity results have been established over the last 20 years, leading to classes of group actions for which the orbit equivalence relation entirely determines the group and its action. We refer to [191, 94, 97] for a comprehensive overview of measured group theory. On the other hand there have so far only been relatively few orbit equivalence "flexibility" results for nonamenable groups. Two results of this kind have been obtained recently by Lewis Bowen in [40, 41]. In [40] Bowen proved that two Bernoulli actions $\mathbb{F}_n \curvearrowright X_0^{\mathbb{F}_n}$ and $\mathbb{F}_n \curvearrowright X_1^{\mathbb{F}_n}$ of the same free group $\mathbb{F}_n$, but with different base probability spaces, are always orbit equivalent. Note that this is a nontrivial result because Bowen proved earlier in [39] that these Bernoulli actions can only be isomorphic if the base probability spaces (X_0, μ_0) and (X_1, μ_1) have the same entropy.

Two free ergodic p.m.p. actions $\Gamma_i \curvearrowright (X_i, \mu_i)$ are called stably orbit equivalent if their orbit equivalence relations can be restricted to non-negligible measurable subsets $\mathcal{U}_i \subset X_i$ such that the resulting equivalence relations on $\mathcal{U}_0$ and $\mathcal{U}_1$ become isomorphic. The number $\mu_1(\mathcal{U}_1)/\mu_0(\mathcal{U}_0)$ is called the compression constant of the stable orbit equivalence. In [41] Bowen proved that the Bernoulli actions $\mathbb{F}_n \curvearrowright X_0^{\mathbb{F}_n}$ and $\mathbb{F}_m \curvearrowright X_1^{\mathbb{F}_m}$ of two different free groups are stably orbit equivalent with compression constant $(n-1)/(m-1)$.

The first aim of this chapter is to give an elementary proof for the above two theorems of Bowen. The concrete stable orbit equivalence that we obtain between $\mathbb{F}_n \curvearrowright X_0^{\mathbb{F}_n}$ and $\mathbb{F}_m \curvearrowright X_1^{\mathbb{F}_m}$ is identical to the one discovered by Bowen. The difference between the two approaches is however the following: rather than writing an explicit formula for the stable orbit equivalence, we construct actions of $\mathbb{F}_n$ and $\mathbb{F}_m$ on (subsets of) the same space, having the same orbits and satisfying an abstract characterization of the Bernoulli action.

Secondly our simpler methods also yield a new orbit equivalence flexibility (actually isomorphism) result that we explain now. Combining the work of many hands [99, 115, 98] it was shown in [81] that every nonamenable group admits uncountably many non orbit equivalent actions (see [109] for a survey). Nevertheless it is still an open problem to give a concrete construction producing such an uncountable family. For a while it has been speculated that for any given nonamenable group Γ the actions

$$\Big\{\Gamma \curvearrowright K^\Gamma/K \;\Big|\; K \text{ a compact second countable group acting by diagonal translation on } K^\Gamma \Big\} \quad (3.1)$$

are non orbit equivalent for nonisomorphic K. Indeed, in [175, Proposition 5.6] it was shown that this is indeed the case whenever every 1-cocycle for the Bernoulli action $\Gamma \curvearrowright K^\Gamma$ with values in either a countable or a compact group $\mathcal{G}$ is cohomologous to a group homomorphism from Γ to $\mathcal{G}$. By Popa's cocycle superrigidity theorems [168, 171], this is the case when Γ contains an infinite normal subgroup with the relative property (T) or when Γ can be written as the direct product of an infinite group and a nonamenable group. Conjecturally the same is true whenever the first ℓ^2-Betti number of Γ vanishes (cf. [158]).

In the last section of this chapter we disprove the above speculation whenever $\Gamma = \Lambda_1 * \cdots * \Lambda_n$ is the free product of n infinite amenable groups, in particular when $\Gamma = \mathbb{F}_n$. We prove that for these Γ and for every compact second countable group K the action $\Gamma \curvearrowright K^\Gamma/K$ is isomorphic with a Bernoulli action of Γ. As we shall see, the special case $\Gamma = \mathbb{F}_n$ is a very easy generalization of [152, Appendix C.(b)] where the same result is proven for $K = \mathbb{Z}/2\mathbb{Z}$ and $\Gamma = \mathbb{F}_2$.

More generally, denote by $\mathcal{G}$ the class of countably infinite groups Γ for which the action $\Gamma \curvearrowright K^\Gamma/K$ is isomorphic with a Bernoulli action of Γ. Then by [152] the class $\mathcal{G}$ contains all infinite amenable groups. We prove in Theorem 3.5.2 that $\mathcal{G}$ is stable under taking free products. By the results cited above, $\mathcal{G}$ does not contain groups that admit an infinite normal subgroup with the relative property (T) and $\mathcal{G}$ does not contain groups that can be written as the direct product of an infinite group and a nonamenable group. So it is a very intriguing problem which groups belong to $\mathcal{G}$.

Terminology and notations

A measure preserving action $\Gamma \curvearrowright (X,\mu)$ of a countable group Γ on a standard probability space (X,μ) is called *essentially free* if a.e. $x \in X$ has a trivial stabilizer and is called *ergodic* if the only Γ-invariant measurable subsets of X have measure 0 or 1. Two free ergodic probability measure preserving (p.m.p.) actions $\Gamma \curvearrowright (X,\mu)$ and $\Lambda \curvearrowright (Y,\eta)$ are called

- *conjugate,* if there exists an isomorphism of groups $\delta : \Gamma \to \Lambda$ and an isomorphism of probability spaces $\Delta : X \to Y$ such that $\Delta(g \cdot x) = \delta(g) \cdot \Delta(x)$ for all $g \in \Gamma$ and a.e. $x \in X$;

- *orbit equivalent,* if there exists an isomorphism of probability spaces $\Delta : X \to Y$ such that $\Delta(\Gamma \cdot x) = \Lambda \cdot \Delta(x)$ for a.e. $x \in X$;
- *stably orbit equivalent,* if there exists a nonsingular isomorphism $\Delta : \mathcal{U} \to \mathcal{V}$ between non-negligible measurable subsets $\mathcal{U} \subset X$ and $\mathcal{V} \subset Y$ such that $\Delta(\Gamma \cdot x \cap \mathcal{U}) = \Lambda \cdot \Delta(x) \cap \mathcal{V}$ for a.e. $x \in \mathcal{U}$. Such a Δ automatically scales the measure by the constant $\eta(\mathcal{V})/\mu(\mathcal{U})$, called the *compression constant* of the stable orbit equivalence.

We say that two p.m.p. actions $\Gamma \curvearrowright (X_i, \mu_i)$ of the same group are *isomorphic* if they are conjugate w.r.t. the identity isomorphism $\mathrm{id} : \Gamma \to \Gamma$, i.e. if there exists an isomorphism of probability spaces $\Delta : X_0 \to X_1$ such that $\Delta(g \cdot x) = g \cdot \Delta(x)$ for all $g \in \Gamma$ and a.e. $x \in X_0$.

Recall that for every countable group Γ and standard probability space (X_0, μ_0), the Bernoulli action of Γ with base space (X_0, μ_0) is the action $\Gamma \curvearrowright X_0^\Gamma$ on the infinite product X_0^Γ equipped with the product probability measure, given by $(g \cdot x)_h = x_{hg}$ for all $g, h \in \Gamma$ and $x \in X_0^\Gamma$. If Γ is an infinite group and (X_0, μ_0) is not reduced to a single atom of mass 1, then $\Gamma \curvearrowright X_0^\Gamma$ is essentially free and ergodic.

Statement of the main results

We first give an elementary proof for the following theorem of Lewis Bowen.

Theorem 3.A (Bowen [41, 40])**.** *For fixed n and varying base probability space (X_0, μ_0) the Bernoulli actions $\mathbb{F}_n \curvearrowright X_0^{\mathbb{F}_n}$ are orbit equivalent.*

If also n varies, the Bernoulli actions $\mathbb{F}_n \curvearrowright X_0^{\mathbb{F}_n}$ and $\mathbb{F}_m \curvearrowright Y_0^{\mathbb{F}_m}$ are stably orbit equivalent with compression constant $(n-1)/(m-1)$.

Next we study factors of Bernoulli actions and prove the following result.

Theorem 3.B. *If $\Gamma = \Lambda_1 * \cdots * \Lambda_n$ is the free product of n infinite amenable groups and if K is a nontrivial second countable compact group equipped with its normalized Haar measure, then the factor action $\Gamma \curvearrowright K^\Gamma/K$ of the Bernoulli action $\Gamma \curvearrowright K^\Gamma$ by the diagonal translation action of K is isomorphic with a Bernoulli action of Γ. In particular, keeping n fixed and varying the Λ_i and K, all the actions $\Gamma \curvearrowright K^\Gamma/K$ are orbit equivalent.*

In the particular case where $\Gamma = \mathbb{F}_n$, the action $\Gamma \curvearrowright K^\Gamma/K$ is isomorphic with the Bernoulli action $\Gamma \curvearrowright (K \times \cdots \times K)^\Gamma$ whose base space is an n-fold direct product of copies of K.

3.2 Preliminaries

Let (X,μ) and (Y,η) be standard probability spaces. We call Δ a *probability space isomorphism* between (X,μ) and (Y,η) if Δ is a measure preserving Borel bijection between conegligible subsets of X and Y. We call Δ a *nonsingular isomorphism* if Δ is a null set preserving Borel bijection between conegligible subsets of X and Y.

Given a sequence of standard probability spaces (X_n,μ_n), we consider the *infinite product* $X = \prod_n X_n$ equipped with the infinite product measure μ. Then, (X,μ) is a standard probability space. The coordinate maps $\pi_n : X \to X_n$ are measure preserving and independent. Moreover, the Borel σ-algebra on X is the smallest σ-algebra such that all π_n are measurable.

Conversely, assume that (Y,η) is a standard probability space and that $\theta_n : Y \to X_n$ is a sequence of Borel maps. Then, the following two statements are equivalent.

1. There exists an isomorphism of probability spaces $\Delta : Y \to X$ such that $\pi_n(\Delta(y)) = \theta_n(y)$ for a.e. $y \in Y$.
2. The maps θ_n are measure preserving and independent, and the σ-algebra on Y generated by the maps θ_n equals the entire Borel σ-algebra of Y up to null sets.

The proof of this equivalence is standard: if the θ_n satisfy the conditions in 2, one defines $\Delta(y)_n := \theta_n(y)$.

Assume that $\Gamma \curvearrowright (X,\mu)$ and $\Lambda \curvearrowright (Y,\eta)$ are essentially free ergodic p.m.p. actions. Assume that $\Delta : X \to Y$ is an orbit equivalence. By essential freeness, we obtain the a.e. well defined Borel map $\omega : \Gamma \times X \to \Lambda$ determined by

$$\Delta(g \cdot x) = \omega(g,x) \cdot \Delta(x) \quad \text{for all } g \in \Gamma \text{ and a.e. } x \in X .$$

Then, ω is a 1-*cocycle* for the action $\Gamma \curvearrowright (X,\mu)$ with values in the group Λ. In general, whenever $\mathcal{G}$ is a Polish group and $\Gamma \curvearrowright (X,\mu)$ is a p.m.p. action, we call a Borel map $\omega : \Gamma \times X \to \mathcal{G}$ a 1-cocycle if ω satisfies

$$\omega(gh,x) = \omega(g,h\cdot x)\,\omega(h,x) \quad \text{for all } g,h \in \Gamma \text{ and a.e. } x \in X .$$

Two 1-cocycles $\omega,\omega' : \Gamma \times X \to \mathcal{G}$ are called *cohomologous* if there exists a Borel map $\varphi : X \to \mathcal{G}$ such that

$$\omega'(g,x) = \varphi(g\cdot x)\,\omega(g,x)\,\varphi(x)^{-1} \quad \text{for all } g \in \Gamma \text{ and a.e. } x \in X .$$

Also a *stable orbit equivalence* gives rise to a 1-cocycle, as follows. So assume that $\Gamma \curvearrowright (X,\mu)$ and $\Lambda \curvearrowright (Y,\eta)$ are essentially free ergodic p.m.p. actions and that $\Delta : \mathcal{U} \to \mathcal{V}$ is a nonsingular isomorphism between the nonnegligible subsets $\mathcal{U} \subset X$ and $\mathcal{V} \subset Y$, such that $\Delta(\mathcal{U} \cap \Gamma \cdot x) = \mathcal{V} \cap \Lambda \cdot \Delta(x)$ for a.e. $x \in \mathcal{U}$. To define the Zimmer 1-cocycle $\omega : \Gamma \times X \to \Lambda$, one first uses the ergodicity of $\Gamma \curvearrowright (X,\mu)$ to choose a Borel map $p : X \to \mathcal{U}$ satisfying $p(x) \in \Gamma \cdot x$ for a.e. $x \in X$. Then, $\omega : \Gamma \times X \to \Lambda$ is uniquely defined such that

$$\Delta(p(g \cdot x)) = \omega(g,x) \cdot \Delta(p(x)) \quad \text{for all } \; g \in \Gamma \; \text{ and a.e. } \; x \in X \; .$$

One checks easily that ω is a 1-cocycle and that, up to cohomology, ω does not depend on the choice of $p : X \to \mathcal{U}$.

We often use 1-cocycles for p.m.p. actions $\Gamma \curvearrowright (X,\mu)$ of a free product group $\Gamma = \Gamma_1 * \Gamma_2$. Given 1-cocycles $\omega_i : \Gamma_i \times X \to \mathcal{G}$, one checks easily that there is a unique 1-cocycle $\omega : \Gamma \times X \to \mathcal{G}$, up to equality a.e., satisfying $\omega(g,x) = \omega_i(g,x)$ for all $g \in \Gamma_i$ and a.e. $x \in X$.

3.3 Orbit equivalence of co-induced actions

Let $\Lambda \curvearrowright (X,\mu)$ be a p.m.p. action. Assume that $\Lambda < G$ is a subgroup. The co-induced action of $\Lambda \curvearrowright X$ to G is defined as follows. Choose a map $r : G \to \Lambda$ such that $r(\lambda g) = \lambda r(g)$ for all $g \in G, \lambda \in \Lambda$ and such that $r(e) = e$. Note that the choice of such a map r is equivalent to the choice of a section $\theta : \Lambda\backslash\Gamma \to \Gamma$ satisfying $\theta(\Lambda e) = e$. Indeed, the formula $g = r(g)\,\theta(\Lambda g)$ provides the correspondence between θ and r.

Once we have chosen $r : G \to \Lambda$, we can define a 1-cocycle $\Omega : \Lambda\backslash G \times G \to \Lambda$ for the right action of G on $\Lambda\backslash G$, given by $\Omega(\Lambda k, g) = r(k)^{-1} r(kg)$ for all $g,k \in G$.

Classically, whenever $\omega : G \times X \to \Lambda$ is a 1-cocycle for an action of G on X, we can induce an action $\Lambda \curvearrowright Y$ to an action $G \curvearrowright X \times Y$ given by $g \cdot (x,y) = (g \cdot x, \omega(g,x) \cdot y)$.

The co-induced action is defined by a similar formula. So assume that $\Lambda \curvearrowright (X,\mu)$ is a p.m.p. action and that $\Lambda < G$ is a subgroup. Choose $r : G \to \Lambda$ with the associated 1-cocycle $\Omega : \Lambda\backslash G \times G \to \Lambda$, as above. Then the formula

$$G \curvearrowright X^{\Lambda\backslash G} \quad \text{where} \quad (g \cdot y)_{\Lambda k} = \Omega(\Lambda k, g) \cdot y_{\Lambda k g}$$

yields a well defined action of G on the product probability space $X^{\Lambda\backslash G}$. It is easy to check that $G \curvearrowright X^{\Lambda\backslash G}$ is a p.m.p. action and that $(\lambda \cdot y)_{\Lambda e} = \lambda \cdot y_{\Lambda e}$ for all $\lambda \in \Lambda$ and $y \in X^{\Lambda\backslash G}$. A different choice of $r : G \to \Lambda$ leads to a cohomologous 1-cocycle Ω and hence an isomorphic action.

Given a subgroup $\Lambda < G$, a subset $I \subset G$ is called a *right transversal* of $\Lambda < G$ if $I \cap \Lambda g$ is a singleton for every $g \in G$.

Up to isomorphism the co-induced action can be characterized as the unique p.m.p. action $G \curvearrowright Y$ for which there exists a measure preserving map $\rho : Y \to X$ with the following properties.

1. $\rho(\lambda \cdot y) = \lambda \cdot \rho(y)$ for all $\lambda \in \Lambda$ and a.e. $y \in Y$.
2. The factor maps $y \mapsto \rho(g \cdot y)$, $g \in G$, generate the Borel σ-algebra on Y, up to null sets.
3. If $I \subset G$ is a right transversal of $\Lambda < G$, then the maps $y \mapsto \rho(g \cdot y)$, $g \in I$, are independent.

To prove this characterization, first observe that the co-induced action satisfies properties 1, 2 and 3 in a canonical way, with $\rho(y) = y_{\Lambda e}$. Conversely assume that $G \curvearrowright Y$ satisfies these properties. Fix a right transversal $I \subset G$ for $\Lambda < G$, with $e \in I$. Combining properties 1 and 2, we see that the factor maps $y \mapsto \rho(g \cdot y)$, $g \in I$, generate the Borel σ-algebra on Y, up to null sets. A combination of property 3 and the characterization of product probability spaces in Section 3.2 then provides the isomorphism of probability spaces $\Delta : Y \to X^{\Lambda \backslash G}$ given by $\Delta(y)_{\Lambda g} = \rho(g \cdot y)$ for all $y \in Y$, $g \in I$. The right transversal $I \subset G$ for $\Lambda < G$ allows to uniquely define the map $r : G \to \Lambda$ such that $r(\lambda g) = \lambda$ for all $\lambda \in \Lambda$ and $g \in I$. This choice of r provides a formula for the co-induced action $G \curvearrowright X^{\Lambda \backslash G}$. It is easy to check that $\Delta(g \cdot y) = g \cdot \Delta(y)$ for all $g \in G$ and a.e. $y \in Y$.

Remark 3.3.1.
1. The above characterization of the co-induced action yields the following result that we use throughout the chapter: the co-induction of the Bernoulli action $\Lambda \curvearrowright (X_0, \mu_0)^\Lambda$ is isomorphic with the Bernoulli action $G \curvearrowright (X_0, \mu_0)^G$. Indeed, the Bernoulli action $G \curvearrowright (X_0, \mu_0)^G$, together with the canonical factor map $X_0^G \to X_0^\Lambda$, satisfies the above characterization of the co-induced action.

2. In certain cases, for instance if $G = \Gamma * \Lambda$, there exists a group homomorphism $\pi : G \to \Lambda$ satisfying $\pi(\lambda) = \lambda$ for all $\lambda \in \Lambda$. Then $r : G \to \Lambda$ can be taken equal to π and the co-induced action $G \curvearrowright X^{\Lambda \backslash G}$ is of the form $(g \cdot y)_{\Lambda k} = \pi(g) \cdot y_{\Lambda kg}$ for all $g, k \in G$ and $y \in X^{\Lambda \backslash G}$.

3. We often make use of diagonal actions: if $\Lambda \curvearrowright (X, \mu)$ and $\Lambda \curvearrowright (Y, \eta)$ are p.m.p. actions, we consider the diagonal action $\Lambda \curvearrowright X \times Y$ given by $\lambda \cdot (x, y) = (\lambda \cdot x, \lambda \cdot y)$. We make the following simple observation: if $\Lambda < G$ and if we denote by $G \curvearrowright \widetilde{X}$, resp. $G \curvearrowright \widetilde{Y}$, the co-induced actions of $\Lambda \curvearrowright X$, resp.

$\Lambda \curvearrowright Y$, to G, then the co-induced action of the diagonal action $\Lambda \curvearrowright X \times Y$ to G is precisely the diagonal action $G \curvearrowright \widetilde{X} \times \widetilde{Y}$.

4. Assume that $\Lambda \curvearrowright (X, \mu)$ is a p.m.p. action and that $\Lambda < G$ is a subgroup. Denote by $G \curvearrowright Y$ the co-induced action and by $\rho : Y \to X$ the canonical Λ-equivariant factor map. Whenever $\Delta_0 : X \to X$ is a p.m.p. automorphism that commutes with the Λ-action, there is a unique p.m.p. automorphism $\Delta : Y \to Y$, up to equality a.e., that commutes with the G-action and such that $\rho(\Delta(y)) = \Delta_0(\rho(y))$ for a.e. $y \in Y$. Writing $Y = X^{\Lambda\backslash\Gamma}$, the automorphism Δ is just the diagonal product of copies of Δ_0. Later we use this easy observation to canonically lift a p.m.p. action $K \curvearrowright (X, \mu)$ of a compact group K, commuting with the Λ-action, to a p.m.p. action $K \curvearrowright Y$ that commutes with the G-action. Moreover, ρ becomes $(\Lambda \times K)$-equivariant. Writing $Y = X^{\Lambda\backslash\Gamma}$, the action $K \curvearrowright Y$ is the diagonal K-action.

We prove that orbit equivalence is preserved under co-induction to a free product. We actually show that the preservation is "K-equivariant" in a precise way that will be needed in the proof of Theorem 3.B. The case where $K = \{e\}$, i.e. co-induction from Λ to $\Gamma * \Lambda$, is due to Lewis Bowen [41]. Recall that similarly as in the case of countable groups, a p.m.p. action $G \curvearrowright (X, \mu)$ of a second countable locally compact group G is called essentially free if a.e. $x \in X$ has a trivial stabilizer (cf. Lemma 3.5.3 in the appendix).

Theorem 3.3.2. *Let Λ_0, Λ_1 and Γ be countable groups and K a compact second countable group. Assume that $\Lambda_i \times K \curvearrowright (X_i, \mu_i)$ are essentially free p.m.p. actions. Denote $G_i := \Gamma * \Lambda_i$ and denote by $G_i \curvearrowright Y_i$ the co-induced action of $\Lambda_i \curvearrowright X_i$ to G_i, together with the natural actions $K \curvearrowright Y_i$ that commute with $G_i \curvearrowright Y_i$ (see Remark 3.3.1.4).*

- *If the actions $\Lambda_i \curvearrowright X_i/K$ are orbit equivalent, then the actions $G_i \curvearrowright Y_i/K$ are orbit equivalent.*
- *If the actions $\Lambda_i \curvearrowright X_i/K$ are conjugate w.r.t. the group isomorphism $\delta : \Lambda_0 \to \Lambda_1$, then the actions $G_i \curvearrowright Y_i/K$ are conjugate w.r.t. the group isomorphism $\mathrm{id} * \delta : G_0 \to G_1$.*

Proof. We start by proving the first item of the theorem.

Let $\Delta_0 : X_0/K \to X_1/K$ be an orbit equivalence between the actions $\Lambda_i \curvearrowright X_i/K$. Denote by $x \mapsto \overline{x}$ the factor map from X_i to X_i/K. Since K acts essentially freely on X_i and K is compact, Lemma 3.5.3 in the appendix provides measurable maps $\theta_i : X_i \to K$ satisfying $\theta_i(k \cdot x) = k\theta_i(x)$ a.e. and such that

$$\Theta_i : X_i \to K \times X_i/K : x \mapsto (\theta_i(x), \overline{x})$$

is a measure preserving isomorphism. Defining $\Delta := \Theta_1^{-1} \circ (\mathrm{id} \times \Delta_0) \circ \Theta_0$, we have found a measure preserving isomorphism $\Delta : X_0 \to X_1$ that is K-equivariant and satisfies $\Delta((\Lambda_0 \times K) \cdot x) = (\Lambda_1 \times K) \cdot \Delta(x)$ for a.e. $x \in X_1$. Using this Δ we may assume that Λ_0, Λ_1 and K act on the same probability space (X, μ) such that the K-action commutes with both the Λ_i-actions and such that $(\Lambda_0 \times K) * x = (\Lambda_1 \times K) \cdot x$ for a.e. $x \in X$. Here and in what follows, we denote the action of $\Lambda_0 \times K$ by $*$ and the action of $\Lambda_1 \times K$ by $\cdot$. We have $k * x = k \cdot x$ for all $k \in K$ and a.e. $x \in X$.

Write $Y = X^{\Lambda_1 \backslash \Gamma * \Lambda_1}$ and denote by $\cdot$ the co-induced action $G_1 \curvearrowright Y$ of $\Lambda_1 \curvearrowright X$ to G_1. Also denote by $\cdot$ the diagonal action $K \curvearrowright Y$, which commutes with $G_1 \curvearrowright Y$. Define the $(\Lambda_1 \times K)$-equivariant factor map $\rho : Y \to X : \rho(y) = y_{\Lambda_1 e}$.

Define the Zimmer 1-cocycles

$$\eta : \Lambda_0 \times X \to \Lambda_1 \times K : \eta(\lambda_0, x) \cdot x = \lambda_0 * x \quad \text{for a.e. } x \in X_1, \lambda_0 \in \Lambda_0 ,$$

$$\eta' : \Lambda_1 \times X \to \Lambda_0 \times K : \eta'(\lambda_1, x) * x = \lambda_1 \cdot x \quad \text{for a.e. } x \in X_1, \lambda_1 \in \Lambda_1 .$$

Since the Λ_0-action commutes with the K-action on X, we have that

$$\eta(\lambda_0, k * x) = k\eta(\lambda_0, x)k^{-1} \quad \text{for all } k \in K, \lambda_0 \in \Lambda_0 \text{ and a.e. } x \in X . \tag{3.2}$$

We define a new action $G_0 \curvearrowright Y$ denoted by $*$ and determined by

$$\gamma * y = \gamma \cdot y \text{ for } \gamma \in \Gamma, y \in Y \quad \text{and} \quad \lambda_0 * y = \eta(\lambda_0, \rho(y)) \cdot y \text{ for } \lambda_0 \in \Lambda_0, y \in Y .$$

Because of (3.2), the action $G_0 \curvearrowright Y$ commutes with $K \curvearrowright Y$.

Define $\omega : G_0 \times Y \to G_1 \times K$ as the unique 1-cocycle for the action $G_0 \overset{*}{\curvearrowright} Y$ satisfying $\omega(\gamma, y) = \gamma$ for all $\gamma \in \Gamma$ and $\omega(\lambda_0, y) = \eta(\lambda_0, \rho(y))$ for all $\lambda_0 \in \Lambda_0$. Then the equality $g * y = \omega(g, y) \cdot y$ holds when $g \in \Gamma$ and when $g \in \Lambda_0$. So the same equality holds for all $g \in G_0$ and a.e. $y \in Y$. In particular $G_0 * \overline{y} \subset G_1 \cdot \overline{y}$ for a.e. $\overline{y} \in Y/K$.

Define $\omega' : G_1 \times Y \to G_0 \times K$ as the unique 1-cocycle satisfying $\omega'(\gamma, y) = \gamma$ for all $\gamma \in \Gamma$ and $\omega'(\lambda_1, y) = \eta'(\lambda_1, \rho(y))$ for all $\lambda_1 \in \Lambda_1$. As above, it follows that $g \cdot y = \omega'(g, y) * y$ for all $g \in G_1$ and a.e. $y \in Y$. Hence, $G_1 \cdot \overline{y} \subset G_0 * \overline{y}$ for a.e. $\overline{y} \in Y/K$. We already proved the converse inclusion so that $G_1 \cdot \overline{y} = G_0 * \overline{y}$ for a.e. $y \in Y/K$.

We prove now that the action $G_0 \overset{*}{\curvearrowright} Y$ together with the Λ_0-equivariant factor map $\rho : Y \to X$ satisfies the abstract characterization for the co-induced action of $\Lambda_0 \curvearrowright X$ to G_0. Once this is proven, the theorem follows because ρ is moreover K-equivariant and the action $G_0 \curvearrowright Y$ commutes with the $K \curvearrowright Y$ (see Remark 3.3.1.4).

We first need to prove that the maps $y \mapsto \rho(g * y)$ are independent and identically distributed when g runs through a right transversal of $\Lambda_0 \subset G_0$. If $g \in G_i = \Gamma * \Lambda_i$, denote by $|g|$ the number of letters from $\Gamma - \{e\}$ that appear in a reduced expression of g. By convention, put $|g| = 0$ if $g \in \Lambda_i$. Define the subsets $I_n \subset G_0$ given by $I_0 := \{e\}$ and

$$I_n := \big\{ g \in G_0 \,\big|\, |g| = n \text{ and the leftmost letter of a reduced expression of } g \text{ belongs to } \Gamma - \{e\} \big\}\,. \quad (3.3)$$

Similarly define $J_n \subset G_1$ and note that $\bigcup_{n=0}^{\infty} J_n$ is a right transversal for $\Lambda_1 < \Gamma * \Lambda_1$. So, in the construction of the co-induced action, we can choose the Λ_1-equivariant map $r : G_1 \to \Lambda_1$ such that $r(g) = e$ for all $g \in J_n$ and all $n \in \mathbb{N}$. Hence $(g \cdot y)_{\Lambda_1 e} = y_{\Lambda_1 g}$ for all $g \in J_n$, $n \in \mathbb{N}$ and a.e. $y \in Y$. For $j \in \Lambda_1 \backslash G_1$ we put $|j| = n$ if $j = \Lambda_1 g$ with $g \in J_n$.

Denote $\omega(g, y) = (\omega_1(g, y), \omega_K(g, y))$ with $\omega_1(g, y) \in G_1$ and $\omega_K(g, y) \in K$. Similarly write $\eta(\lambda, x) = (\eta_1(\lambda, x), \eta_K(\lambda, x))$. Note that for $\lambda \in \Lambda_0 - \{e\}$ we have $\eta_1(\lambda, x) \neq e$ for a.e. $x \in X$. Indeed, if $\eta_1(\lambda, x) = e$ for a fixed $\lambda \in \Lambda_0 - \{e\}$, then the element $(\lambda, \eta_K(\lambda, x)^{-1})$ of $\Lambda_0 \times K$ stabilizes x and the essential freeness of $\Lambda_0 \times K \curvearrowright X$ implies that this can only happen for x belonging to a negligible subset of X. One then proves easily by induction on n that

- for a.e. $y \in Y$ and all $n \in \mathbb{N}$, the map $g \mapsto \omega_1(g, y)$ is a bijection of I_n onto J_n,
- for all $n \in \mathbb{N}, g \in I_n$, the map $y \mapsto \omega(g, y)$ only depends on the coordinates y_j, $|j| \leqslant n - 1$.

Since for all $g \in I_n$ we have $\omega_1(g, y) \in J_n$, it follows that

$$\rho(g * y) = (g * y)_{\Lambda_1 e} = (\omega(g, y) \cdot y)_{\Lambda_1 e} = \omega_K(g, y) \cdot y_{\Lambda_1 \omega_1(g,y)} \quad (3.4)$$

for all $n \in \mathbb{N}$, $g \in I_n$ and a.e. $y \in Y$. We now use Lemma 3.3.4 to prove that for all $n \in \mathbb{N}$, the set $\{y \mapsto \rho(g * y) \mid g \in I_n\}$ forms a family of independent random variables that are independent of the coordinates y_j, $|j| \leqslant n - 1$, and that only depend on the coordinates y_j, $|j| \leqslant n$. More concretely, we write $\mathcal{J}_n = \{\Lambda_1 g \mid |g| \leqslant n\}$ and we apply Lemma 3.3.4 to the countable set $\mathcal{J}_n - \mathcal{J}_{n-1}$, the direct product

$$Z := X^{\mathcal{J}_{n-1}} \times X^{\mathcal{J}_n - \mathcal{J}_{n-1}}$$

and the family of measurable maps $\omega_g : Z \to K \times (\mathcal{J}_n - \mathcal{J}_{n-1})$ indexed by $g \in I_n$, only depending on the coordinates y_j, $j \in \mathcal{J}_{n-1}$ and given by

$$\omega_g : y \mapsto (\omega_K(g, y), \Lambda_1 \omega_1(g, y))\,.$$

Since $g \mapsto \omega_1(g,y)$ is a bijection of I_n onto J_n, we have that $g \mapsto \Lambda_1\omega_1(g,y)$ is a bijection of I_n onto $\mathcal{J}_n - \mathcal{J}_{n-1}$. A combination of Lemma 3.3.4 and formula (3.4) then implies that $\{y \mapsto \rho(g * y) \mid g \in I_n\}$ is a family of independent random variables that are independent of the coordinates y_j, $j \in \mathcal{J}_{n-1}$. By construction, these random variables only depend on the coordinates y_j, $|j| \leqslant n$. Having proven these statements for all $n \in \mathbb{N}$, it follows that $\{y \mapsto \rho(g * y) \mid g \in \bigcup_n I_n\}$ is a family of independent random variables.

Denote by $\mathcal{B}_0$ the smallest σ-algebra on Y such that $Y \to X_1 : y \mapsto \rho(g * y)$ is $\mathcal{B}_0$-measurable for all $g \in G_0$. It remains to prove that $\mathcal{B}_0$ is the entire σ-algebra of Y. Note that by construction, the map $Y \to Y : y \mapsto g * y$ is $\mathcal{B}_0$-measurable for all $g \in G_0$. Since ρ is K-equivariant and the actions $K \curvearrowright Y$ and $G_0 \curvearrowright Y$ commute, we also get that the map $y \mapsto k * y$ is $\mathcal{B}_0$-measurable for every $k \in K$. We must prove that $y \mapsto y_i$ is $\mathcal{B}_0$-measurable for every $n \in \mathbb{N}$ and $i \in \Lambda_1 \backslash G_1$ with $|i| = n$. This follows by induction on n, because for all $g \in J_n$ we have

$$y_{\Lambda_1 g} = \rho(g \cdot y) = \rho(\omega'(g,y) * y)$$

and because $y \mapsto \omega'(g,y)$ only depends on the coordinates y_j, $|j| \leqslant n-1$.

To prove the second item of the theorem, it suffices to make the following observation. If the actions $\Lambda_i \curvearrowright X_i/K$ are conjugate w.r.t. the isomorphism $\delta : \Lambda_0 \to \Lambda_1$, then in the proof of the first item, the Zimmer 1-cocycle η is of the form $\eta(\lambda_0, x) = (\delta(\lambda_0), \eta_K(\lambda_0, x))$. So the 1-cocycle $\omega : G_0 \times Y \to G_1 \times K$ is of the form $\omega(g,y) = ((\mathrm{id} * \delta)(g), \omega_K(g,y))$. This immediately implies that the actions $G_i \curvearrowright Y_i/K$ are conjugate w.r.t. the isomorphism $\mathrm{id} * \delta$. □

Corollary 3.3.3 (Bowen [41])**.** *For fixed n and varying base probability space (X_0, μ_0) the Bernoulli actions $\mathbb{F}_n \curvearrowright X_0^{\mathbb{F}_n}$ are orbit equivalent.*

Proof. By Remark 3.3.1.1, the co-induction of a Bernoulli action is again a Bernoulli action over the same base space. Let X_0 and X_1 be nontrivial base probability spaces. By Dye's theorem [69], the Bernoulli actions $\mathbb{Z} \curvearrowright X_0^{\mathbb{Z}}$ and $\mathbb{Z} \curvearrowright X_1^{\mathbb{Z}}$ are orbit equivalent. By Theorem 3.3.2 their co-induced actions to $\mathbb{F}_n = \mathbb{F}_{n-1} * \mathbb{Z}$ are orbit equivalent. But these co-induced actions are isomorphic to the Bernoulli actions $\mathbb{F}_n \curvearrowright X_i^{\mathbb{F}_n}$. □

We used the following easy independence lemma.

Lemma 3.3.4. *Let (X, μ) and (X_0, μ_0) be standard probability spaces and let $H \curvearrowright (X_0, \mu_0)$ be a measure preserving action. Let I be a countable set. Consider $Z = X \times X_0^I$ with the product probability measure. Assume that $\mathcal{F}$ is a family of measurable maps $\omega : Z \to H \times I$. Write $\omega(x,y) = (\omega_1(x,y), \omega_2(x,y))$. Assume that*

- *for almost every* $z \in Z$*, the map* $\mathcal{F} \to I : \omega \mapsto \omega_2(z)$ *is injective,*
- *for every* $\omega \in \mathcal{F}$*, the map* $z \mapsto \omega(z)$ *only depends on the variable* $Z \to X : (x, y) \mapsto x$*.*

Then, $\{(x, y) \mapsto \omega_1(x, y) \cdot y_{\omega_2(x,y)} \mid \omega \in \mathcal{F}\}$ *is a family of independent identically* (X_0, μ_0)*-distributed random variables that are independent of* $(x, y) \mapsto x$*.*

Proof. Since the maps $\omega \in \mathcal{F}$ only depend on the variable $(x, y) \mapsto x$, we view $\omega \in \mathcal{F}$ as a map from X to $H \times I$. We have to prove that $\{(x, y) \mapsto \omega_1(x) \cdot y_{\omega_2(x)} \mid \omega \in \mathcal{F}\}$ is a family of independent identically (X_0, μ_0)-distributed random variables that are independent of $(x, y) \mapsto x$. But conditioning on $x \in X$, we get that the variables

$$X_0^I \to X_0 : y \mapsto \omega_1(x) \cdot y_{\omega_2(x)}$$

are independent and (X_0, μ_0)-distributed because the coordinates $\omega_2(x)$, for $\omega \in \mathcal{F}$, are distinct elements of I and because the action $H \curvearrowright X_0$ is measure preserving. So the lemma is proven. □

3.4 Stable orbit equivalence of Bernoulli actions

Denote by a, b the standard generators of $\mathbb{F}_2$. Denote by $\langle a \rangle$ and $\langle b \rangle$ the subgroups of $\mathbb{F}_2$ generated by a, resp. b. Let (X_0, μ_0) be a standard probability space and consider the Bernoulli action $\mathbb{F}_2 \curvearrowright X_0^{\mathbb{F}_2}$ given by $(g \cdot x)_h = x_{hg}$.

Whenever (X_0, μ_0) is a probability space, the Bernoulli action $\Gamma \curvearrowright X_0^\Gamma$ can be characterized up to isomorphism as the unique p.m.p. action $\Gamma \curvearrowright X$ for which there exists a factor map $\pi : X \to X_0$ such that the maps $x \mapsto \pi(g \cdot x)$, $g \in \Gamma$, are independent and generate, up to null sets, the whole σ-algebra of X.

We prove the stable orbit equivalence of Bernoulli actions as a combination of the following three lemmas. Fix $\kappa \in \mathbb{N}$, $\kappa \geqslant 2$, and denote $X_0 = \{0, \ldots, \kappa - 1\}$ equipped with the uniform probability measure. Let (Y_0, η_0) be any standard probability space (that is not reduced to a single atom). Denote by $r : \mathbb{F}_2 \to \mathbb{Z}/\kappa\mathbb{Z}$ the group morphism determined by $r(a) = 0$ and $r(b) = 1$. Identify X_0 with $\mathbb{Z}/\kappa\mathbb{Z}$ and denote by $\cdot$ the action of $\mathbb{Z}/\kappa\mathbb{Z}$ on X_0 given by addition in $\mathbb{Z}/\kappa\mathbb{Z}$.

Lemma 3.4.1. *Consider the action* $\mathbb{F}_2 \curvearrowright X := X_0^{\langle b \rangle \backslash \mathbb{F}_2}$ *given by* $(g \cdot x)_{\langle b \rangle h} = r(g) \cdot x_{\langle b \rangle hg}$*. Let* $\mathbb{F}_2 \curvearrowright Y_0^{\mathbb{F}_2}$ *be the Bernoulli action. Then the diagonal action* $\mathbb{F}_2 \curvearrowright X \times Y_0^{\mathbb{F}_2}$ *given by* $g \cdot (x, y) = (g \cdot x, g \cdot y)$ *is orbit equivalent with a Bernoulli action of* $\mathbb{F}_2$*.*

Lemma 3.4.2. *The action $\mathbb{F}_2 \curvearrowright X$ defined in Lemma 3.4.1 is stably orbit equivalent with compression constant $1/\kappa$ with a Bernoulli action of $\mathbb{F}_{1+\kappa}$.*

Lemma 3.4.3. *Let $\Gamma \curvearrowright (X, \mu)$ be any free ergodic p.m.p. action of an infinite group Γ. Assume that $\kappa \in \mathbb{N}$ and that $\Gamma \curvearrowright X$ is stably orbit equivalent with compression constant $1/\kappa$ with a Bernoulli action of some countable group Λ. Let (Y_0, η_0) be any standard probability space and $\Gamma \curvearrowright Y_0^\Gamma$ the Bernoulli action. Then also the diagonal action $\Gamma \curvearrowright X \times Y_0^\Gamma$ is stably orbit equivalent with compression constant $1/\kappa$ with a Bernoulli action of Λ.*

Proof of Theorem 3.A

We already deduce Theorem 3.A from the above three lemmas.

Proof of Theorem 3.A. We first prove that Lemmas 3.4.1, 3.4.2, 3.4.3 yield a Bernoulli action of $\mathbb{F}_2$ that is stably orbit equivalent with compression constant $1/\kappa$ with a Bernoulli action of $\mathbb{F}_{1+\kappa}$. Indeed, by Lemma 3.4.1 a Bernoulli action of $\mathbb{F}_2$ is orbit equivalent with the diagonal action $\mathbb{F}_2 \curvearrowright X \times Y_0^{\mathbb{F}_2}$. By Lemma 3.4.2, the action $\mathbb{F}_2 \curvearrowright X$ is stably orbit equivalent with compression constant $1/\kappa$ with a Bernoulli action of $\mathbb{F}_{1+\kappa}$. But then, Lemma 3.4.3 says that the same holds for the diagonal action $\mathbb{F}_2 \curvearrowright X \times Y_0^{\mathbb{F}_2}$.

Combined with Corollary 3.3.3 it follows that *all* Bernoulli actions of $\mathbb{F}_2$ are stably orbit equivalent with all Bernoulli actions of $\mathbb{F}_m$, $m \geqslant 2$, with compression constant $1/(m-1)$. By transitivity of stable orbit equivalence, all Bernoulli actions of $\mathbb{F}_n$ and $\mathbb{F}_m$ are stably orbit equivalent with compression constant $(n-1)/(m-1)$. □

Proof of Lemma 3.4.1

Proof of Lemma 3.4.1. View $\mathbb{Z}$ as the subgroup of $\mathbb{F}_2$ generated by b. Let $\mathbb{Z} \curvearrowright Y_0^\mathbb{Z}$ be the Bernoulli action. Consider the action $\mathbb{Z} \curvearrowright X_0 \times Y_0^\mathbb{Z}$ given by $g \cdot (x, y) = (r(g) \cdot x, g \cdot y)$. Note that $\mathbb{Z} \curvearrowright X_0 \times Y_0^\mathbb{Z}$ is a free ergodic p.m.p. action. Using Remark 3.3.1 (statements 1, 2 and 3), one gets that the action $\mathbb{F}_2 \curvearrowright X \times Y_0^{\mathbb{F}_2}$ given in the formulation of Lemma 3.4.1 is precisely the co-induction of $\mathbb{Z} \curvearrowright X_0 \times Y_0^\mathbb{Z}$ to $\mathbb{F}_2$. By Dye's theorem [69], the free ergodic p.m.p. action $\mathbb{Z} \curvearrowright X_0 \times Y_0^\mathbb{Z}$ is orbit equivalent with a Bernoulli action of $\mathbb{Z}$. By Remark 3.3.1.1, the co-induction of the latter is a Bernoulli action of $\mathbb{F}_2$. So by Theorem 3.3.2, the action $\mathbb{F}_2 \curvearrowright X \times Y_0^{\mathbb{F}_2}$ is orbit equivalent with a Bernoulli action of $\mathbb{F}_2$. □

Proof of Lemma 3.4.2

Proof of Lemma 3.4.2. We have $X = X_0^{\langle b\rangle\backslash\mathbb{F}_2}$ and the action $\mathbb{F}_2 \curvearrowright X$ is given by $(g\cdot x)_{\langle b\rangle h} = r(g)\cdot x_{\langle b\rangle hg}$. Write $Z = X_0^{\mathbb{Z}}$ and denote by $\rho : X \to Z$ the factor map given by $\rho(x)_n = x_{\langle b\rangle a^n}$. Denote by $\cdot$ the Bernoulli action $\mathbb{Z} \curvearrowright Z$ and note that $\rho(a^n \cdot x) = n\cdot\rho(x)$ for all $x \in X$ and $n \in \mathbb{Z}$.

Define the subsets V_i, $i = 0,\dots,\kappa-1$, of Z given by $V_i := \{z \in Z \mid z_0 = i\}$. Similarly define $W_i \subset X$ given by $W_i = \rho^{-1}(V_i)$. Note that W_0 has measure $1/\kappa$. To prove the lemma we define a p.m.p. action of $\mathbb{F}_{1+\kappa}$ on W_0 such that $\mathbb{F}_{1+\kappa} * x = \mathbb{F}_2\cdot x \cap W_0$ for a.e. $x \in W_0$ and such that $\mathbb{F}_{1+\kappa} \curvearrowright W_0$ is a Bernoulli action.

By Dye's theorem [69], there exists a Bernoulli action $\mathbb{Z} \overset{*}{\curvearrowright} V_0$ such that $\mathbb{Z} * z = \mathbb{Z}\cdot z \cap V_0$ for a.e. $z \in V_0$. Denote by $\eta : \mathbb{Z}\times V_0 \to \mathbb{Z}$ the corresponding 1-cocycle for the $*$-action determined by $n * z = \eta(n,z)\cdot z$ for $n \in \mathbb{Z}$ and a.e. $z \in V_0$.

Since the Bernoulli action $\mathbb{Z} \overset{\cdot}{\curvearrowright} Z$ is ergodic and since all the subsets $V_i \subset Z$ have the same measure, we can choose measure preserving isomorphisms $\alpha_i : V_0 \to V_i$ satisfying $\alpha_i(z) \in \mathbb{Z}\cdot z$ for a.e. $z \in \mathbb{Z}$ and take α_0 to be the identity isomorphism. Let $\varphi_i^0 : V_0 \to \mathbb{Z}$ and $\psi_i^0 : V_i \to \mathbb{Z}$ be the maps determined by $\alpha_i(z) = \varphi_i^0(z)\cdot z$ for a.e. $z \in V_0$ and $\alpha_i^{-1}(z) = \psi_i^0(z)\cdot z$ for a.e. $z \in V_i$. Define the corresponding measure preserving isomorphisms $\theta_i : W_0 \to W_i$ given by $\theta_i(x) = \varphi_i(x)\cdot x$ and $\theta_i^{-1}(x) = \psi_i(x)\cdot x$ where $\varphi_i(x) = a^{\varphi_i^0(\rho(x))}$ and $\psi_i(x) = a^{\psi_i^0(\rho(x))}$.

Denote by a and b_i, $i = 0,\dots,\kappa-1$, the generators of $\mathbb{F}_{1+\kappa}$. Define the p.m.p. action $\mathbb{F}_{1+\kappa} \overset{*}{\curvearrowright} W_0$ given by

$$a^n * x = a^{\eta(n,\rho(x))}\cdot x \quad\text{and}\quad b_i * x = \theta_{i+1}^{-1}(b\cdot\theta_i(x)) \quad\text{for all}\;\; x \in W_0\;.$$

Note that the action is well defined: if $x \in W_0$, then $\theta_i(x) \in W_i$ and hence $b\cdot\theta_i(x) \in W_{i+1}$. We use the convention that $W_\kappa = W_0$ and $\theta_\kappa = \mathrm{id}$. Observe that $\rho(a^n * x) = n * \rho(x)$ for all $n \in \mathbb{Z}$ and a.e. $x \in W_0$.

It remains to prove that $\mathbb{F}_{1+\kappa} * x = \mathbb{F}_2\cdot x \cap W_0$ for a.e. $x \in W_0$ and that $\mathbb{F}_{1+\kappa} \curvearrowright W_0$ is a Bernoulli action.

Denote by $\omega : \mathbb{F}_{1+\kappa}\times W_0 \to \mathbb{F}_2$ the unique 1-cocycle for the $*$-action determined by

$$\omega(a^n,x) = a^{\eta(n,\rho(x))}$$

and

$$\omega(b_i,x) = \psi_{i+1}(b\cdot\theta_i(x))\,b\,\varphi_i(x)\;.$$

By construction, the formula $g * x = \omega(g,x)\cdot x$ holds for all $g \in \{a, b_0,\dots,b_{\kappa-1}\}$ and a.e. $x \in W_0$. Since ω is a 1-cocycle for the action $\mathbb{F}_{1+\kappa} \overset{*}{\curvearrowright} W_0$, the same formula holds for all $g \in \mathbb{F}_{1+\kappa}$ and a.e. $x \in W_0$. In particular, $\mathbb{F}_{1+\kappa} * x \subset \mathbb{F}_2 \cdot x \cap W_0$ for a.e. $x \in W_0$. To prove the converse inclusion we define the inverse 1-cocycle for ω.

Define $q_0 : Z \to V_0$ given by $q_0(z) = \alpha_i^{-1}(z)$ when $z \in V_i$. Denote by $\eta' : \mathbb{Z} \times Z \to \mathbb{Z}$ the 1-cocycle for the $\cdot$-action determined by $q_0(n \cdot z) = \eta'(n,z) * q_0(z)$. Whenever $z \in V_0$, we have $z = q_0(z)$ and hence

$$\eta'(\eta(n,z),z) * z = \eta'(\eta(n,z),z) * q_0(z) = q_0(\eta(n,z) \cdot z) = q_0(n * z) = n * z \ . \quad (3.5)$$

Since $*$ is an essentially free action of $\mathbb{Z}$, it follows that $\eta'(\eta(n,z),z) = n$ for all $n \in \mathbb{Z}$ and a.e. $z \in V_0$.

Denote by $\omega' : \mathbb{F}_2 \times X \to \mathbb{F}_{1+\kappa}$ the unique 1-cocycle for the $\cdot$-action determined by

$$\omega'(a^n, x) = a^{\eta'(n,\rho(x))} \quad \text{for } n \in \mathbb{Z} \text{ and a.e. } x \in X,$$

and

$$\omega'(b,x) = b_i \quad \text{for a.e. } x \in W_i.$$

Define $q : X \to W_0$ given by $q(x) = \theta_i^{-1}(x)$ when $x \in W_i$. Note that $\rho(q(x)) = q_0(\rho(x))$ for a.e. $x \in X$. We prove that $q(g \cdot x) = \omega'(g,x) * q(x)$ for all $g \in \mathbb{F}_2$ and a.e. $x \in X$. If $g = a^n$ for some $n \in \mathbb{Z}$, we know that both $q(g \cdot x)$ and $\omega'(g,x) * q(x)$ belong to $\langle a \rangle \cdot x$. So to prove that they are equal, it suffices to check that they have the same image under ρ. The following computation shows that this is indeed the case.

$$\rho(q(a^n \cdot x)) = q_0(\rho(a^n \cdot x)) = q_0(n \cdot \rho(x)) = \eta'(n,\rho(x)) * q_0(\rho(x)) \ ,$$

while

$$\begin{aligned} \rho(\omega'(a^n,x) * q(x)) &= \rho(a^{\eta'(n,\rho(x))} * q(x)) \\ &= \eta'(n,\rho(x)) * \rho(q(x)) = \eta'(n,\rho(x)) * q_0(\rho(x)) \ . \end{aligned}$$

Since by definition of the action $*$ we have that $b_i * \theta_i^{-1}(x) = \theta_{i+1}^{-1}(b \cdot x)$ whenever $x \in W_i$, the formula $\omega'(g,x) * q(x) = q(g \cdot x)$ also holds when $g = b$. Hence, the same formula holds for all $g \in \mathbb{F}_2$ and a.e. $x \in X$. In particular, $\mathbb{F}_2 \cdot x \cap W_0 \subset \mathbb{F}_{1+\kappa} * x$ for a.e. $x \in W_0$. The converse inclusion was already proven above. Hence, $\mathbb{F}_{1+\kappa} * x = \mathbb{F}_2 \cdot x \cap W_0$ for a.e. $x \in W_0$.

Denote by $\mathcal{J} \subset \mathbb{F}_{1+\kappa}$ the union of $\{e\}$ and all the reduced words that start with one of the letters $b_i^{\pm 1}$, $i = 0, \ldots, \kappa - 1$. Note that $\mathcal{J}$ is a right transversal for $\langle a \rangle < \mathbb{F}_{1+\kappa}$. It remains to prove that

$$\{W_0 \to V_0 : x \mapsto \rho(g * x) \mid g \in \mathcal{J}\}$$

is a family of independent random variables that generate, up to null sets, the whole σ-algebra on W_0. Indeed, we already know that $\mathbb{Z} \overset{*}{\curvearrowright} V_0$ is a Bernoulli action so that it will follow that $\mathbb{F}_{1+\kappa} \curvearrowright W_0$ is the co-induction of a Bernoulli action, hence a Bernoulli action itself (see Remark 3.3.1.1).

We equip both $\mathbb{F}_2$ and $\mathbb{F}_{1+\kappa}$ with a length function. For $g \in \mathbb{F}_2$ we denote by $|g|$ the number of letters $b^{\pm 1}$ appearing in the reduced expression of g, while for $g \in \mathbb{F}_{1+\kappa}$ we denote by $|g|$ the number of letters $b_i^{\pm 1}$, $i = 0, \ldots, \kappa - 1$, appearing in the reduced expression of g. By induction on the length of g, one easily checks that $|\omega(g,x)| \leqslant |g|$ for all $g \in \mathbb{F}_{1+\kappa}$ and a.e. $x \in W_0$, and that $|\omega'(g,x)| \leqslant |g|$ for all $g \in \mathbb{F}_2$ and a.e. $x \in X$.

We next claim that

$$\omega'(\omega(g,x),x) = g \quad \text{for all } g \in \mathbb{F}_{1+\kappa} \text{ and a.e. } x \in W_0. \tag{3.6}$$

Once this claim is proven, it follows that $|\omega(g,x)| = |g|$ for all $g \in \mathbb{F}_{1+\kappa}$ and a.e. $x \in W_0$: indeed, the strict inequality $|\omega(g,x)| < |g|$ would lead to the contradiction

$$|g| = |\omega'(\omega(g,x),g)| \leqslant |\omega(g,x)| < |g| \; .$$

First note that for $g = a^n$ formula (3.6) follows immediately from (3.5). So it remains to prove (3.6) when $g = b_i$. First observe that $\omega'(\varphi_i(x),x) = e$ for a.e. $x \in W_0$. Indeed,

$$\omega'(\varphi_i(x),x) * x = q(\varphi_i(x) \cdot x) = q(\theta_i(x)) = x$$

and since the $*$-action of $\langle a \rangle$ on W_0 is essentially free, it follows that $\omega'(\varphi_i(x),x) = e$. Similarly, $\omega'(\psi_i(x),x) = e$ for a.e. $x \in W_i$. Take $x \in W_0$ and write $x' := b\varphi_i(x) \cdot x$. Note that $x' = b \cdot \theta_i(x)$ and that $x' \in W_{i+1}$. So,

$$\begin{aligned}\omega'(\omega(b_i,x),x) &= \omega'(\psi_{i+1}(x')\, b\, \varphi_i(x),x) \\ &= \omega'(\psi_{i+1}(x'),x')\, \omega'(b,\theta_i(x))\, \omega'(\varphi_i(x),x) = e\, b_i\, e = b_i \; .\end{aligned}$$

So (3.6) holds for $g = a^n$ and $g = b_i$. Hence (3.6) holds for all $g \in \mathbb{F}_{1+\kappa}$. Note that (3.6) implies that the action $\mathbb{F}_{1+\kappa} \overset{*}{\curvearrowright} W_0$ is essentially free. Indeed, if $g \in \mathbb{F}_{1+\kappa}$, $x \in W_0$ and $g * x = x$, it follows that $\omega(g,x) \cdot x = x$. Since the $\cdot$-action is essentially free, we conclude that $\omega(g,x) = e$. But then by (3.6)

$$g = \omega'(\omega(g,x),x) = \omega'(e,x) = e \; .$$

Define the subsets $\mathcal{C}(n) \subset \langle b \rangle \backslash \mathbb{F}_2$ given by $\mathcal{C}(n) := \{\langle b \rangle g \mid g \in \mathbb{F}_2, |g| \leqslant n\}$. Also define $\mathcal{J}_n := \{g \in \mathcal{J} \mid |g| \leqslant n\}$. We prove by induction on n that the following two statements hold.

1_n. If $g \in \mathbb{F}_{1+\kappa}$ and $|g| \leqslant n$, then $x \mapsto \omega(g,x)$ only depends on the coordinates x_i, $i \in \mathcal{C}(n)$.

2_n. The set $\{W_0 \to V_0 \mid x \mapsto \rho(g * x) \mid g \in \mathcal{J}_n\}$ is a family of independent random variables that only depend on the coordinates x_i, $i \in \mathcal{C}(n)$.

Since e is the only element in $\mathcal{J}$ of length 0, statements 1_0 and 2_0 are trivial. Assume that statements 1_n and 2_n hold for a given n.

Any element in $\mathbb{F}_{1+\kappa}$ of length $n+1$ can be written as a product gh with $|g| = 1$ and $|h| = n$. By the cocycle equality, we have

$$\omega(gh,x) = \omega(g, h * x)\, \omega(h,x) = \omega(g, \omega(h,x) \cdot x)\, \omega(h,x) \; .$$

By statement 1_n, we know that the map $x \mapsto \omega(g,x)$ only depends on the coordinates x_i, $i \in \mathcal{C}(1)$, and that the map $x \mapsto \omega(h,x)$ only depends on on the coordinates x_i, $i \in \mathcal{C}(n)$. So, $x \mapsto \omega(gh,x)$ only depends on the coordinates x_i, $i \in \mathcal{C}(n)$, and the map

$$x \mapsto (\omega(h,x) \cdot x)_{\langle b \rangle k} = r(\omega(h,x)) \cdot x_{\langle b \rangle k \omega(h,x)} \quad \text{for } \; |k| \leqslant 1 \; .$$

Again by statement 1_n these maps only depend on the coordinates x_i, $i \in \mathcal{C}(n{+}1)$, so that statement 1_{n+1} is proven.

Define, for $i = 0, \dots, \kappa - 1$ and $\varepsilon = \pm 1$,

$$\mathcal{J}_n^{i,\varepsilon} := \left\{ g \in \mathbb{F}_{1+\kappa} \;\middle|\; |g| = n \; \text{ and } \; |b_i^\varepsilon g| = n+1 \right\} .$$

It follows that

$$\mathcal{J}_{n+1} = \mathcal{J}_n \cup \bigcup_{i \in \{0,\dots,\kappa-1\}, \varepsilon \in \{\pm 1\}} b_i^\varepsilon \, \mathcal{J}_n^{i,\varepsilon} \; .$$

Since we assumed that statement 2_n holds, in order to prove statement 2_{n+1}, it suffices to show that

$$\{x \mapsto \rho(b_i^\varepsilon g * x) \mid i = 0, \dots, \kappa - 1, \varepsilon = \pm 1, g \in \mathcal{J}_n^{i,\varepsilon}\}$$

is a family of independent random variables that only depend on the coordinates x_i, $i \in \mathcal{C}(n+1)$, and that are independent of the coordinates x_i, $i \in \mathcal{C}(n)$.

Note that $\rho(b_i g * x) = \alpha_{i+1}^{-1}(\rho(b \cdot \theta_i(g * x)))$ while $\rho(b_i^{-1} g * x) = \alpha_i^{-1}(\rho(b^{-1} \cdot \theta_{i+1}(g * x)))$. The value of $\rho(b \cdot \theta_i(g * x))$ at 0 is constantly equal to $i+1$, while the value of $\rho(b^{-1} \cdot \theta_{i+1}(g * x))$ at 0 is constantly equal to i. Therefore we have to prove that

$$\begin{aligned} &\{x \mapsto \rho(b \cdot \theta_i(g * x))_m \mid i = 0, \dots, \kappa - 1, g \in \mathcal{J}_n^{i,+}, m \in \mathbb{Z} - \{0\}\} \\ &\cup \{x \mapsto \rho(b^{-1} \cdot \theta_{i+1}(g * x))_m \mid i = 0, \dots, \kappa - 1, g \in \mathcal{J}_n^{i,-}, m \in \mathbb{Z} - \{0\}\} \end{aligned} \quad (3.7)$$

is a family of independent random variables that only depend on the coordinates x_i, $i \in \mathcal{C}(n+1)$, and that are independent of the coordinates x_i, $i \in \mathcal{C}(n)$.

Write

$$\omega_i^\varepsilon(g,x) := \begin{cases} b\,\varphi_i(g * x)\,\omega(g,x) & \text{if } \varepsilon = 1, \\ b^{-1}\,\varphi_{i+1}(g * x)\,\omega(g,x) & \text{if } \varepsilon = -1. \end{cases}$$

The random variables in (3.7) are precisely equal to

$$\{x \mapsto r(\omega_i^\varepsilon(g,x)) \cdot x_{\langle b \rangle a^m \omega_i^\varepsilon(g,x)} \mid \\ i = 0, \ldots, \kappa - 1, \varepsilon = \pm 1, g \in \mathcal{J}_n^{i,\varepsilon}, m \in \mathbb{Z} - \{0\}\} \; . \quad (3.8)$$

So we have to prove that (3.8) is a family of independent random variables that only depend on the coordinates x_i, $i \in \mathcal{C}(n+1)$, and that are independent of the coordinates x_i, $i \in \mathcal{C}(n)$. By statement 1_n, the maps $x \mapsto \omega_i^\varepsilon(g,x)$, and in particular $x \mapsto r(\omega_i^\varepsilon(g,x))$, only depend on the coordinates x_i, $i \in \mathcal{C}(n)$. So, we have to prove that

$$\{x \mapsto x_{\langle b \rangle a^m \omega_i^\varepsilon(g,x)} \mid i = 0, \ldots, \kappa - 1, \varepsilon = \pm 1, g \in \mathcal{J}_n^{i,\varepsilon}, m \in \mathbb{Z} - \{0\}\} \; . \quad (3.9)$$

is a family of independent random variables that only depend on the coordinates x_i, $i \in \mathcal{C}(n+1)$, and that are independent of the coordinates x_i, $i \in \mathcal{C}(n)$.

We apply Lemma 3.3.4 to the countable set $\mathcal{C}(n+1) - \mathcal{C}(n)$ and the direct product

$$X_0^{\mathcal{C}(n)} \times X_0^{\mathcal{C}(n+1) - \mathcal{C}(n)} \; .$$

Since the maps $x \mapsto \omega_i^\varepsilon(g,x)$ only depend on the coordinates x_i, $i \in \mathcal{C}(n)$, it remains to check that the cosets $\langle b \rangle a^m \omega_i^\varepsilon(g,x)$ belong to $\mathcal{C}(n+1) - \mathcal{C}(n)$ and that they are distinct for fixed $x \in W_0$ and varying $i \in \{0, \ldots, \kappa - 1\}$, $\varepsilon \in \{\pm 1\}$ and $g \in \mathcal{J}_n^{i,\varepsilon}$.

Note that $\omega(b_i^\varepsilon g, x) \in \langle a \rangle\, \omega_i^\varepsilon(g,x)$. Hence,

$$|\omega_i^\varepsilon(g,x)| = |\omega(b_i^\varepsilon g, x)| = |b_i^\varepsilon g| = n+1$$

because $g \in \mathcal{J}_n^{i,\varepsilon}$. Since $|\omega(g,x)| = n$ and $|\omega_i^\varepsilon(g,x)| = n+1$, it follows from the defining formula of ω_i^ε that the first letter of $\omega_i^\varepsilon(g,x)$ must be b^ε. So the first letter of $a^m \omega_i^\varepsilon(g,x)$, $m \neq 0$, is $a^{\pm 1}$. This implies that $\langle b \rangle a^m \omega_i^\varepsilon(g,x)$ belongs to $\mathcal{C}(n+1) - \mathcal{C}(n)$. It also follows that if

$$\langle b \rangle a^m \omega_i^\varepsilon(g,x) = \langle b \rangle a^{m'} \omega_{i'}^{\varepsilon'}(g',x) \; ,$$

then we must have $m = m'$, $\varepsilon = \varepsilon'$ and $\omega_i^\varepsilon(g,x) = \omega_{i'}^{\varepsilon'}(g',x)$. Assume $\varepsilon = \varepsilon' = 1$, the other case being analogous. So,

$$\varphi_i(g * x)\,\omega(g,x) = \varphi_{i'}(g',x)\,\omega(g',x) \; .$$

Applying these elements to x, it follows that $\theta_i(g * x) = \theta_{i'}(g' * x)$. Since the ranges of θ_i and $\theta_{i'}$ are disjoint for $i \neq i'$, it follows that $i = i'$. So, $g * x = g' * x$. Since we have seen above that the action $\mathbb{F}_{1+\kappa} \overset{*}{\curvearrowright} W_0$ is essentially free, it follows that $g = g'$.

We have proven that (3.8) is a family of independent random variables that only depend on the coordinates x_i, $i \in \mathcal{C}(n+1)$, and that are independent of the coordinates x_i, $i \in \mathcal{C}(n)$. So, statement 2_{n+1} holds.

To conclude the proof of the lemma, it remains to show that the random variables $x \mapsto \rho(g * x)$, $g \in \mathbb{F}_{1+\kappa}$, generate up to null sets the whole σ-algebra of W_0. Denote by $\mathcal{B}_0$ the σ-algebra on W_0 generated by these random variables. By construction, $x \mapsto g * x$ is $\mathcal{B}_0$-measurable for every $g \in \mathbb{F}_{1+\kappa}$. Since $x \mapsto \rho(x)$ is $\mathcal{B}_0$-measurable, the formula

$$q(a^n \cdot x) = a^{\eta'(n,\rho(x))} * x$$

shows that $x \mapsto q(a^n \cdot x)$ is $\mathcal{B}_0$-measurable for every $n \in \mathbb{Z}$. Denote by $\mathcal{B}_1$ the smallest σ-algebra on X containing $\mathcal{B}_0$, containing the subsets $W_0, \ldots, W_{\kappa-1} \subset X$ and making $q : X \to W_0$ a $\mathcal{B}_1$-measurable map. Note that the restriction of $\mathcal{B}_1$ to W_0 equals $\mathcal{B}_0$ and that $\mathcal{U} \subset X$ is $\mathcal{B}_1$-measurable if and only if $q(\mathcal{U} \cap W_i)$ is $\mathcal{B}_0$-measurable for every $i = 0, \ldots, \kappa - 1$. It therefore suffices to prove that $\mathcal{B}_1$ is the whole σ-algebra of X. By construction, $\rho : X \to Z$ is $\mathcal{B}_1$-measurable and by the above, also $x \mapsto a^n \cdot x$ is $\mathcal{B}_1$-measurable for every $n \in \mathbb{Z}$. If $x \in W_i$, we have that $b \cdot x = \theta_{i+1}^{-1}(b_i * \theta_i(x))$ and it follows that $x \mapsto b \cdot x$ is $\mathcal{B}_1$-measurable. Hence, $x \mapsto g \cdot x$ is $\mathcal{B}_1$-measurable for every $g \in \mathbb{F}_2$. Since ρ is $\mathcal{B}_1$-measurable, it follows that $x \mapsto x_{\langle b \rangle g}$ is $\mathcal{B}_1$-measurable for every $g \in \mathbb{F}_2$. Hence $\mathcal{B}_1$ is the entire product σ-algebra. □

Proof of Lemma 3.4.3

Proof of Lemma 3.4.3. We denote by a dot $\cdot$ the action of Γ on X. Let $X_1 \subset X$ be a subset of measure $1/\kappa$. We are given a p.m.p. action $\Lambda \overset{*}{\curvearrowright} X_1$ such that $\Lambda * x = \Gamma \cdot x \cap X_1$ for a.e. $x \in X_1$ and such that $\Lambda \curvearrowright X_1$ is isomorphic with a Λ-Bernoulli action. This means that we have a probability space U and a factor map $\pi : X_1 \to U$ such that the random variables $\{x \mapsto \pi(\lambda * x) \mid \lambda \in \Lambda\}$ are independent, identically distributed and generating the Borel σ-algebra of X_1. Denote by $\omega : \Lambda \times X_1 \to \Gamma$ the 1-cocycle determined by $\omega(\lambda, x) \cdot x = \lambda * x$ for all $\lambda \in \Lambda$ and a.e. $x \in X_1$. Put $Y = Y_0^\Gamma$ and define the action $\Lambda \curvearrowright X_1 \times Y$ given by

$$\lambda * (x, y) = \omega(\lambda, x) \cdot (x, y) = (\lambda * x, \omega(\lambda, x) \cdot y) .$$

By construction, $\Lambda * (x,y) \subset \Gamma \cdot (x,y) \cap X_1 \times Y$. But also the converse inclusion holds. Indeed, if we have $\gamma \in \Gamma$, $x \in X_1$ and $y \in Y$ such that $\gamma \cdot x \in X_1$, we can take $\lambda \in \Lambda$ such that $\lambda * x = \gamma \cdot x$. Hence $\omega(\lambda, x) = \gamma$ and also $\gamma \cdot (x,y) = \lambda * (x,y)$.

It remains to prove that $\Lambda \curvearrowright X_1 \times Y$ is isomorphic with a Λ-Bernoulli action.

By ergodicity of $\Gamma \curvearrowright X$, choose a partition (up to measure zero) $X = X_1 \sqcup \cdots \sqcup X_\kappa$ with $\mu(X_i) = 1/\kappa$ and choose measurable maps $\varphi_i : X_1 \to \Gamma$ such that the formulae $\theta_i(x) = \varphi_i(x) \cdot x$ define measure space isomorphisms $\theta_i : X_1 \to X_i$. Take $\varphi_1(x) = e$ for all $x \in X_1$. Define the measurable map

$$\rho : X_1 \times Y \to U \times Y_0^\kappa : \rho(x,y) = \left(\pi(x), y_{\varphi_1(x)}, \ldots, y_{\varphi_\kappa(x)}\right) .$$

We prove that ρ is measure preserving and that the random variables $\{(x,y) \mapsto \rho(\lambda * (x,y)) \mid \lambda \in \Lambda\}$ are independent, identically distributed and generating the Borel σ-algebra of $X_1 \times Y$.

We first claim that for a.e. $x \in X_1$

$$\mathcal{F} := \Big(\varphi_i(\lambda * x)\omega(\lambda, x)\Big)_{\lambda \in \Lambda \text{ and } i = 1, \ldots, \kappa} \tag{3.10}$$

is an enumeration of Γ without repetitions. Observe that

$$\varphi_i(\lambda * x)\omega(\lambda, x) \cdot x = \theta_i(\lambda * x) .$$

It follows that $\mathcal{F} \cdot x = \Gamma \cdot x$. Since $\Gamma \curvearrowright X$ is essentially free, it follows that $\mathcal{F}$ enumerates the whole of Γ. If $\varphi_i(\lambda * x)\omega(\lambda, x) = \varphi_j(\lambda' * x)\omega(\lambda', x)$, it follows that $\theta_i(\lambda * x) = \theta_j(\lambda' * x)$. For $i \neq j$, the sets X_i and X_j are disjoint. So, $i = j$ and $\lambda * x = \lambda' * x$. Being a Bernoulli action of an infinite group, $\Lambda \overset{*}{\curvearrowright} X_1$ is essentially free and we conclude that $\lambda = \lambda'$. This proves the claim.

Since for a.e. $x \in X_1$ the elements $\varphi_1(x), \ldots, \varphi_\kappa(x)$ are distinct, it follows from Lemma 3.3.4 that the random variables $(x,y) \mapsto \pi(x)$ and $(x,y) \mapsto y_{\varphi_i(x)}$, $i = 1, \ldots, \kappa$, are all independent. Since they are all measure preserving as well, we conclude that ρ is measure preserving. Note that

$$\rho(\lambda * (x,y)) = \left(\pi(\lambda * x), y_{\varphi_1(\lambda * x)\omega(\lambda, x)}, \ldots, y_{\varphi_\kappa(\lambda * x)\omega(\lambda, x)}\right) .$$

It therefore remains to prove that

$$\{(x,y) \mapsto \pi(\lambda * x) \mid \lambda \in \Lambda\} \cup \{(x,y) \mapsto y_{\varphi_i(\lambda * x)\omega(\lambda, x)} \mid \lambda \in \Lambda, i = 1, \ldots, \kappa\}$$

is an independent family of random variables that generate, up to null sets, the Borel σ-algebra of $X_1 \times Y$. The factor map π was chosen in such a way that the random variables $\{x \mapsto \pi(\lambda * x) \mid \lambda \in \Lambda\}$ are independent and generate, up to null sets, the Borel σ-algebra of X_1. So, we must prove that

$$\{(x,y) \mapsto y_{\varphi_i(\lambda * x)\omega(\lambda, x)} \mid \lambda \in \Lambda, i = 1, \ldots, \kappa\} \tag{3.11}$$

forms a family of independent random variables that are independent of $(x,y) \mapsto x$ and that, together with $(x,y) \mapsto x$, generate up to null sets the Borel σ-algebra of $X_1 \times Y$. We apply Lemma 3.3.4 to the countable set Γ, the direct product $X_1 \times Y_0^\Gamma$ and the family of maps $X_1 \to \Gamma : x \mapsto \varphi_i(\lambda * x)\omega(\lambda, x)$ indexed by $\lambda \in \Lambda, i = 1, \ldots, \kappa$. Since for a.e. $x \in X_1$, the set $\mathcal{F}$ in (3.10) is an enumeration of Γ, it follows from Lemma 3.3.4 that (3.11) is indeed a family of independent random variables that are moreover independent of $(x,y) \mapsto x$.

Denote by $\mathcal{B}_1$ the smallest σ-algebra on $X_1 \times Y$ such that the map $(x,y) \mapsto x$ and the random variables in (3.11) are measurable. It remains to prove that, up to null sets, $\mathcal{B}_1$ is the Borel σ-algebra of $X_1 \times Y$. So, it remains to prove that for all $g \in \Gamma$, the random variables $(x,y) \mapsto y_g$ are $\mathcal{B}_1$-measurable. Put $\mathcal{J} = \{1, \ldots, \kappa\} \times \Lambda$ and define the Borel map $\eta : \mathcal{J} \times X_1 \to \Gamma$ given by $\eta(i, \lambda, x) := \varphi_i(\lambda * x)\omega(\lambda, x)$. Since for a.e. $x \in X_1$, the family $\mathcal{F}$ in (3.10) is an enumeration of Γ, we can take a Borel map $\gamma : \Gamma \times X_1 \to \mathcal{J}$ such that $\eta(\gamma(g,x),x) = g$ for all $g \in \Gamma$ and a.e. $x \in X_1$. By the definition of $\mathcal{B}_1$ and η, we know that the map

$$\mathcal{J} \times X_1 \times Y \to Y_0 : (j,x,y) \mapsto y_{\eta(j,x)} \tag{3.12}$$

is $\mathcal{B}_1$-measurable. Fix $g \in \Gamma$. Since $(x,y) \mapsto x$ is $\mathcal{B}_1$-measurable, also $(x,y) \mapsto (\gamma(g,x), x, y)$ is $\mathcal{B}_1$-measurable. The composition with the map in (3.12) yields $(x,y) \mapsto y_g$ a.e. So $(x,y) \mapsto y_g$ is $\mathcal{B}_1$-measurable. This concludes the proof of the lemma. □

3.5 Isomorphisms of factors of Bernoulli actions of free products

Before proving Theorem 3.B, we need the following elementary lemma.

Lemma 3.5.1. *Let Γ, Λ be countable groups and K a nontrivial second countable compact group equipped with its normalized Haar measure. Consider the action $(\Gamma * \Lambda) \times K \curvearrowright X := K^{\Gamma \backslash \Gamma * \Lambda}$ where $\Gamma * \Lambda$ shifts the indices and K acts by diagonal left translation. The resulting factor action $\Gamma * \Lambda \curvearrowright X/K$ is isomorphic with the co-induced action of $\Lambda \curvearrowright K^\Lambda/K$ to $\Gamma * \Lambda$.*

Proof. Define the factor map $\rho : K^{\Gamma \backslash \Gamma * \Lambda} \to K^\Lambda$ given by $\rho(x)_\lambda = x_{\Gamma\lambda}$. Note that ρ is $(\Lambda \times K)$-equivariant. Denote $X := K^{\Gamma \backslash \Gamma * \Lambda}$ and denote by $x \mapsto \overline{x}$ the factor map of X onto X/K. So we get the Λ-equivariant factor map $\overline{\rho} : X/K \to K^\Lambda/K : \overline{\rho}(\overline{x}) = \overline{\rho(x)}$. We prove that $\Gamma * \Lambda \curvearrowright X/K$ together with $\overline{\rho}$ satisfies the abstract characterization of the co-induced action of $\Lambda \curvearrowright K^\Lambda/K$ to $\Gamma * \Lambda$.

For $g \in \Gamma * \Lambda$, denote by $|g|$ the number of letters from $\Gamma - \{e\}$ appearing in a reduced expression for g. Define the subsets $I_n \subset \Gamma * \Lambda$ given by $I_0 := \{e\}$ and

$$I_n := \bigl\{g \in \Gamma * \Lambda \bigm| |g| = n \text{ and the reduced expression of } g \text{ starts with a letter from } \Gamma - \{e\} \bigr\} .$$

Note that $\bigcup_{n=0}^{\infty} I_n$ is a right transversal for $\Lambda < \Gamma * \Lambda$. So we have to prove that

$$\{\overline{x} \mapsto \overline{\rho}(g \cdot \overline{x}) \mid n \in \mathbb{N}, g \in I_n\} \tag{3.13}$$

is a family of independent random variables that generate, up to null sets, the whole σ-algebra of X/K.

For $i \in \Gamma\backslash\Gamma * \Lambda$, we write $|i| = n$ if $i = \Gamma g$, where $|g| = n$ and the reduced expression for g starts with a letter from $\Lambda - \{e\}$. For every $\lambda \in \Lambda - \{e\}$, define $\theta_\lambda : K^\Lambda/K \to K : \theta_\lambda(\overline{x}) = x_e^{-1} x_\lambda$. Observe that for all $g \in I_n$ and $\lambda \in \Lambda - \{e\}$, we have

$$\theta_\lambda(\overline{\rho}(g \cdot \overline{x})) = x_{\Gamma g}^{-1}\, x_{\Gamma \lambda g} \; . \tag{3.14}$$

Since $g \in I_n$ starts with a letter from $\Gamma - \{e\}$, we have $|\Gamma \lambda g| = |g| = n$, while $|\Gamma g| = n - 1$. Write $\mathcal{I}_n := \{i \in \Gamma\backslash\Gamma * \Lambda \mid |i| \leqslant n\}$. We apply Lemma 3.3.4 to the countable set $\mathcal{I}_n - \mathcal{I}_{n-1}$, the direct product

$$Z := K^{\mathcal{I}_{n-1}} \times K^{\mathcal{I}_n - \mathcal{I}_{n-1}}$$

and the family of maps $\omega_{g,\lambda} : Z \to K \times (\mathcal{I}_n - \mathcal{I}_{n-1})$, indexed by $g \in I_n, \lambda \in \Lambda - \{e\}$, only depending on the coordinates x_i, $i \in \mathcal{I}_{n-1}$, and given by

$$\omega_{g,\lambda} : x \mapsto (x_{\Gamma g}^{-1}, \Gamma \lambda g) \; .$$

Since the elements $\Gamma \lambda g$, for $g \in I_n, \lambda \in \Lambda - \{e\}$, enumerate $\mathcal{I}_n - \mathcal{I}_{n-1}$, it follows from Lemma 3.3.4 that the random variables

$$\{X \to K : x \mapsto x_{\Gamma g}^{-1} x_{\Gamma \lambda g} \mid g \in I_n, \lambda \in \Lambda - \{e\} \}$$

are independent, only depend on the coordinates x_i, $|i| \leqslant n$, and are independent of the coordinates x_i, $|i| \leqslant n - 1$. In combination with (3.14), it follows that (3.13) is indeed a family of independent random variables.

Denote by $\mathcal{B}_0$ the smallest σ-algebra on X/K for which all the functions $\overline{x} \mapsto \overline{\rho}(g \cdot \overline{x})$, $g \in \Gamma * \Lambda$, are $\mathcal{B}_0$-measurable. Formula (3.14) and an induction on n show that $\overline{x} \mapsto x_{\Gamma e}^{-1}\, x_i$ is $\mathcal{B}_0$-measurable for every $i \in \Gamma\backslash\Gamma * \Lambda$ with $|i| \leqslant n$. Hence, $\mathcal{B}_0$ is the entire σ-algebra on X/K. □

Theorem 3.B will be an immediate corollary of the following general result.

Theorem 3.5.2. *Let Γ_i, $i = 0, 1$, be countable groups and K a nontrivial second countable compact group equipped with its normalized Haar measure. Assume that $\Gamma_i \curvearrowright K^{\Gamma_i}/K$ is isomorphic with the Bernoulli action $\Gamma_i \curvearrowright Y_i^{\Gamma_i}$ with base space (Y_i, μ_i). Write $G := \Gamma_0 * \Gamma_1$. Then $G \curvearrowright K^G/K$ is isomorphic with the Bernoulli action $G \curvearrowright (Y_0 \times Y_1)^G$ with base space $Y_0 \times Y_1$.*

Proof. Put $A := K^{\Gamma_0}$ and denote by α the action $\Gamma_0 \times K \overset{\alpha}{\curvearrowright} A$ where Γ_0 shifts the indices and K acts by diagonal left translation. Put $B := Y_0^{\Gamma_0} \times K$ and denote by β the action $\Gamma_0 \times K \overset{\beta}{\curvearrowright} B$ where Γ_0 only acts on the factor $Y_0^{\Gamma_0}$ in a Bernoulli way and K only acts on the factor K by translation. Our assumptions say that $\Gamma_0 \curvearrowright A/K$ and $\Gamma_0 \curvearrowright B/K$ are isomorphic actions. We apply Theorem 3.3.2 to these two actions of Γ_0.

So denote $G = \Gamma_0 * \Gamma_1$ and denote by $G \curvearrowright \tilde{A}$, resp. $G \curvearrowright \tilde{B}$, the co-induced actions of $\Gamma_0 \curvearrowright A$, resp. $\Gamma_0 \curvearrowright B$, to G. Note that these actions come together with natural actions $K \curvearrowright \tilde{A}$ and $K \curvearrowright \tilde{B}$ that commute with G-actions. By Theorem 3.3.2, the actions $G \curvearrowright \tilde{A}/K$ and $G \curvearrowright \tilde{B}/K$ are isomorphic.

We now identify the actions $G \times K \curvearrowright \tilde{A}$ and $G \times K \curvearrowright \tilde{B}$ with the following known actions. First, the action $G \times K \curvearrowright \tilde{A}$ is canonically isomorphic with $G \times K \curvearrowright K^G$ where G acts in a Bernoulli way and K acts by diagonal left translation. Secondly, using Remark 3.3.1.3, the action $G \times K \curvearrowright \tilde{B}$ is isomorphic with the action $G \times K \curvearrowright Y_0^G \times K^{\Gamma_0 \backslash G}$ where G acts diagonally in a Bernoulli way and K only acts on the second factor by diagonal left translation. In combination with the previous paragraph, it follows that the action $G \curvearrowright K^G/K$ is isomorphic with the diagonal action $G \curvearrowright Y_0^G \times (K^{\Gamma_0 \backslash G})/K$.

From Lemma 3.5.1, we know that $G \curvearrowright (K^{\Gamma_0 \backslash G})/K$ is isomorphic with the co-induced action of $\Gamma_1 \curvearrowright K^{\Gamma_1}/K$ to G. Since we assumed that $\Gamma_1 \curvearrowright K^{\Gamma_1}/K$ is isomorphic with the Bernoulli action $\Gamma_1 \curvearrowright Y_1^{\Gamma_1}$, it follows that $G \curvearrowright (K^{\Gamma_0 \backslash G})/K$ is isomorphic with the Bernoulli action $G \curvearrowright Y_1^G$. In combination with the previous paragraph, it follows that $G \curvearrowright K^G/K$ is isomorphic with the Bernoulli action $G \curvearrowright (Y_0 \times Y_1)^G$. □

Proof of Theorem 3.B

Proof of Theorem 3.B. Since the action $\Lambda_i \curvearrowright K^{\Lambda_i}/K$ arises as the factor of a Bernoulli action and Λ_i is amenable, it follows from [152] that $\Lambda_i \curvearrowright K^{\Lambda_i}/K$ is isomorphic with a Bernoulli action $\Lambda_i \curvearrowright Y_i^{\Lambda_i}$. Repeatedly applying Theorem 3.5.2, it follows that $\Gamma \curvearrowright K^{\Gamma}/K$ is isomorphic with the Bernoulli action $\Gamma \curvearrowright (Y_1 \times \cdots \times Y_n)^{\Gamma}$.

The special case $\Gamma = \mathbb{F}_n$ is a very easy generalization of [152, Appendix C.(b)]. Denote by $x \mapsto \overline{x}$ the quotient map from $K^{\mathbb{F}_n}$ to $K^{\mathbb{F}_n}/K$. Denote by $a_1, \ldots, a_n$ the free generators of $\mathbb{F}_n$. Define the measurable map

$$\theta : K^{\mathbb{F}_n}/K \to (K \times \cdots \times K)^{\mathbb{F}_n} : \theta(\overline{x})_g = (x_g^{-1} x_{a_1 g}, \ldots, x_g^{-1} x_{a_n g}) .$$

We shall prove that θ is an isomorphism between $\mathbb{F}_n \curvearrowright K^{\mathbb{F}_n}/K$ and $\mathbb{F}_n \curvearrowright (K \times \cdots \times K)^{\mathbb{F}_n}$. First note that θ is indeed $\mathbb{F}_n$-equivariant. It remains to prove that

$$\{\overline{x} \mapsto x_g^{-1} x_{a_i g} \mid i = 1, \ldots, n \;,\; g \in \mathbb{F}_n\} \tag{3.15}$$

is a family of independent random variables on $K^{\mathbb{F}_n}/K$ that generate up to null sets the whole σ-algebra of $K^{\mathbb{F}_n}/K$. Denote by $|g|$ the word length of an element $g \in \mathbb{F}_n$. Define for $i \in \{1, \ldots, n\}$, $\varepsilon = \pm 1$ and $k \in \mathbb{N}$, the subsets $I_k^{i,\varepsilon} \subset \mathbb{F}_n$ given by

$$I_k^{i,\varepsilon} := \left\{ g \in \mathbb{F}_n \;\middle|\; |g| = k \;,\; |a_i^\varepsilon g| = k+1 \right\} .$$

If $|g| = k$ and $|a_i g| = k-1$, we compose the random variable $\overline{x} \mapsto x_g^{-1} x_{a_i g}$ by the map $K \to K : y \mapsto y^{-1}$ and observe that $a_i g \in I_{k-1}^{i,-1}$. So we need to prove that

$$\{\overline{x} \mapsto x_g^{-1} x_{a_i^\varepsilon g} \mid i = 1, \ldots, n \;,\; \varepsilon = \pm 1 \;,\; k \in \mathbb{N} \;,\; g \in I_k^{i,\varepsilon}\} \tag{3.16}$$

is a family of independent random variables that generate up to null sets the whole σ-algebra of $K^{\mathbb{F}_n}/K$.

Write $\mathcal{I}_k = \{g \in \mathbb{F}_n \mid |g| \leqslant k\}$ and fix $k \in \mathbb{N}$. We apply Lemma 3.3.4 to the countable set $\mathcal{I}_{k+1} - \mathcal{I}_k$, the direct product

$$Z := K^{\mathcal{I}_k} \times K^{\mathcal{I}_{k+1} - \mathcal{I}_k}$$

and the family of maps $\omega_{i,\varepsilon,g} : Z \to K \times (\mathcal{I}_{k+1} - \mathcal{I}_k)$ indexed by the set

$$\mathcal{F} := \{(i, \varepsilon, g) \mid i = 1, \ldots, n \;,\; \varepsilon = \pm 1 \;,\; g \in I_k^{i,\varepsilon}\} ,$$

only depending on the coordinates x_i, $i \in \mathcal{I}_k$, and given by

$$\omega_{i,\varepsilon,g} : x \mapsto (x_g^{-1}, a_i^\varepsilon g) .$$

Since the elements $a_i^\varepsilon g$ with $(i, \varepsilon, g) \in \mathcal{F}$ enumerate $\mathcal{I}_{k+1} - \mathcal{I}_k$, it follows from Lemma 3.3.4 that $\{x \mapsto x_g^{-1} x_{a_i^\varepsilon g} \mid i = 1, \ldots, n \;,\; \varepsilon = \pm 1 \;,\; g \in I_k^{i,\varepsilon}\}$ is a family of independent random variables that are independent of the coordinates x_h, $|h| \leqslant k$. By construction, these random variables only depend on the coordinates x_h, $|h| \leqslant k+1$. This being proven for all $k \in \mathbb{N}$, it follows that (3.16) is a family of independent random variables. Hence the same is true for (3.15). These random variables can be easily seen to generate up to null sets the whole σ-algebra of $K^{\mathbb{F}_n}/K$. □

Appendix: essentially free actions of locally compact groups

A p.m.p. action of a second countable locally compact group G on a standard probability space (X, μ) is an action of the group G on the set X such that $G \times X \to X : (g, x) \mapsto g \cdot x$ is a Borel map and such that for all $g \in G$ and all Borel sets $A \subset X$, we have $\mu(g \cdot A) = \mu(A)$.

For every $x \in X$, we define the subgroup $\operatorname{Stab} x$ of G given by $\operatorname{Stab} x = \{g \in G \mid g \cdot x = x\}$. For the sake of completeness, we give a proof for the following folklore lemma.

Lemma 3.5.3. *Let $G \curvearrowright (X, \mu)$ be a p.m.p. action of a second countable locally compact group G on a standard probability space (X, μ), as above.*

1. *The set $X_0 := \{x \in X \mid \operatorname{Stab} x = \{e\}\ \}$ is a G-invariant Borel subset of X.*
2. *Assume that $\mu(X_0) = 1$ and that G is compact. Denote by m the normalized Haar measure on G. There exists a standard probability space (Y_0, η) and a bijective Borel isomorphism $\theta : G \times Y_0 \to X_0$ such that $\theta(gh, y) = g \cdot \theta(h, y)$ for all $g, h \in G$, $y \in Y_0$, and such that $\theta_*(m \times \eta) = \mu$.*

A p.m.p. action $G \curvearrowright (X, \mu)$ is called essentially free if the Borel set $\{x \in X \mid \operatorname{Stab} x = \{e\}\}$ has measure 1.

Proof. By [220, Theorem 3.2], there exists a continuous action of G on a Polish space Y and an injective Borel map $\psi : X \to Y$ satisfying $\psi(g \cdot x) = g \cdot \psi(x)$ for all $g \in G$ and $x \in X$. Since ψ is injective, $\psi(X)$ is a Borel subset of Y and ψ is a Borel isomorphism of X onto $\psi(X)$ (see e.g. [127, Theorem 15.1]). So, we actually view X as a G-invariant Borel subset of Y.

To prove 1, fix a sequence of compact subsets $K_n \subset G - \{e\}$ such that $G - \{e\} = \bigcup_{n=1}^{\infty} K_n$. Also fix a metric d on Y that induces the topology on Y. Define

$$f_n : X \to \mathbb{R} : f_n(x) = \min_{g \in K_n} d(g \cdot x, x) \; .$$

Whenever $\mathcal{F}_n \subset K_n$ is a countable dense subset, we have $f_n(x) = \inf_{g \in \mathcal{F}_n} d(g \cdot x, x)$, so that f_n is Borel. Since $\operatorname{Stab} x = \{e\}$ if and only if $f_n(x) > 0$ for all n, statement 1 follows.

To prove 2, assume that $\mu(X_0) = 1$ and that G is compact. Since G acts continuously on Y and G is compact, all orbits $G \cdot y$ are closed. By [127, Theorem 12.16], we can choose a Borel subset $Y_1 \subset Y$ such that $Y_1 \cap G \cdot y$ is a singleton for every $y \in Y$. Define $Y_0 := Y_1 \cap X_0$. By construction, the map

$$\theta : G \times Y_0 \to X_0 : \theta(g, y) = g \cdot y$$

is Borel, bijective and satisfies $\theta(gh, y) = g \cdot \theta(h, y)$ for all $g, h \in G$ and $y \in Y_0$. Then also θ^{-1} is Borel (see e.g. [127, Theorem 15.1]). The formula $\eta_0 := (\theta^{-1})_*(\mu)$ yields a G-invariant probability measure on $G \times Y_0$. Defining the probability measure η on Y_0 as the push forward of η_0 under the quotient map $(g, y) \mapsto y$, the G-invariance of η_0 together with the Fubini theorem, imply that $\eta_0 = m \times \eta$. □

Chapter 4

Tensor C*-categories arising as bimodule categories of II$_1$ factors

This chapter is based on our joint work with Sébastien Falguières [85]. We prove that if $\mathcal{C}$ is a tensor C*-category in a certain class, then there exists an uncountable family of pairwise non stably isomorphic II$_1$ factors (M_i) such that the bimodule category of M_i is equivalent to $\mathcal{C}$ for all i. In particular, we prove that every finite tensor C*-category is the bimodule category of a II$_1$ factor. As an application we prove the existence of a II$_1$ factor for which the set of indices of finite index irreducible subfactors is $\left\{1, \frac{5+\sqrt{13}}{2}, 12+3\sqrt{13}, 4+\sqrt{13}, \frac{11+3\sqrt{13}}{2}, \frac{13+3\sqrt{13}}{2}, \frac{19+5\sqrt{13}}{2}, \frac{7+\sqrt{13}}{2}\right\}$. We also give the first example of a II$_1$ factor M such that Bimod(M) is explicitly calculated and has an uncountable number of isomorphism classes of irreducible objects.

4.1 Introduction

The description of symmetries of a II$_1$ factor M, such as the *fundamental group* $\mathcal{F}(M)$ of Murray and von Neumann and the *outer automorphism group* Out(M), is a central and usually very hard problem in the theory of II$_1$ factors. Over the last ten years, Sorin Popa developed his *deformation-rigidity* theory [164, 166, 167] and settled many long standing open problems in this direction.

See [214, 169, 217] for a survey. In particular, he obtained the first complete calculations of fundamental groups [164] and outer automorphism groups [117]. His methods were used in further calculations. Without being exhaustive, see for example [166, 176, 62] concerning fundamental groups and [175, 215, 86] for outer automorphism groups.

Bimodules ${}_M\mathcal{H}_M$ over a II_1 factor M having finite left and right M-dimension are said to be of finite *Jones index* (see [53, 160]) and they give rise to a category, which we denote by $\mathrm{Bimod}(M)$. Endowed with the Connes tensor product of M-M-bimodules, $\mathrm{Bimod}(M)$ is a compact tensor C*-category, in the sense of Longo and Roberts [134].

The bimodule category of a II_1 factor M may be seen as a *generalized symmetry group* of M. It contains a lot of structural information on M and encodes several other invariants of M. Indeed, if $\mathrm{grp}(M)$ denotes the group-like elements in $\mathrm{Bimod}(M)$, i.e. bimodules of index 1, one has the following short exact sequence

$$1 \to \mathrm{Out}(M) \to \mathrm{grp}(M) \to \mathcal{F}(M) \to 1\ .$$

Finite index subfactors $N \subset M$ are also encoded in a certain sense by the bimodule category $\mathrm{Bimod}(M)$, since, denoting $N \subset M \subset M_1$ the *Jones basic construction*, we obtain a finite index bimodule ${}_M\mathrm{L}^2(M_1)_M$.

As explained above, in [117] the first actual computation of the outer automorphism group of II_1 factors was achieved, using a combination of relative property (T) and amalgamated free products. Extending their methods, in [216], Vaes proved the existence of a II_1 factor M with trivial bimodule category. As a consequence, all the symmetry groups and subfactors of M were trivial. Also relying on Popa's methods, in [87], Vaes and the first author proved that the representation category of any compact second countable group can be realized as the bimodule category of a II_1 factor. More precisely, given a compact second countable group G, there exists a II_1 factor M and a minimal action $G \curvearrowright M$ such that, denoting M^G the fixed point II_1 factor, the natural fully faithful tensor functor $\mathrm{Rep}(G) \to \mathrm{Bimod}(M^G)$ is an equivalence of tensor C*-categories. Both papers followed closely [117] and thus, they give only existence results.

Explicit results on the calculation of bimodule categories are obtained in [215] and [64]. Both articles are based on generalizations of Popa's seminal papers [164, 166] on Bernoulli crossed products. In [215], Vaes gave explicit examples of group-measure space II_1 factors M for which the fusion algebra, i.e. isomorphism classes of finite index bimodules and fusion rules, were calculated. The complete calculation of the category of bimodules over II_1 factors coming from [215] was obtained by Deprez and Vaes in [64]. Even more is proven in [64], since the C*-bicategory of II_1 factors commensurable with M, i.e. those II_1 factors N

admitting a finite index N-M-bimodule, is also computed and explicitly arises as the bicategory associated with a Hecke pair of groups.

Note that by [67], representation categories of compact second countable groups can be characterized abstractly as symmetric compact tensor C*-categories with countably many isomorphism classes of irreducible objects. Among compact tensor C*-categories, *finite tensor C*-categories*, i.e. those which admit only finitely many isomorphism classes of irreducible objects, form another natural class. We prove that every finite tensor C*-category arises as the bimodule category of a II_1 factor.

Theorem 4.A. *Let $\mathcal{C}$ be a finite tensor C*-category. Then there is a II_1 factor M such that* $\mathrm{Bimod}(M) \simeq \mathcal{C}$.

As an application of the above theorem, we prove the existence of a II_1 factor for which the set of indices of irreducible finite index subfactors can be explicitly calculated and contains irrationals. Recall the amazing theorem of Jones, proving that the index of an inclusion of II_1 factors $N \subset M$ ranges in the set

$$\mathcal{I} = \left\{ 4\cos\left(\frac{\pi}{n}\right)^2 \mid n = 3, 4, 5, \dots \right\} \cup [4, +\infty] \, .$$

Given a II_1 factor M, Jones defines the invariant

$$\mathcal{C}(M) = \{\lambda \mid \text{there is a finite index irreducible inclusion } N \subset M \text{ of index } \lambda\} \, .$$

Jones proved that every element of $\mathcal{I}$ arises as the index of a not necessarily irreducible subfactor of the hyperfinite II_1 factor. However, the problem of computing $\mathcal{C}(R)$ is still widely open. In [215, 216], Vaes proved the existence of II_1 factors M for which $\mathcal{C}(M) = \{1\}$ and $\mathcal{C}(M) = \{n^2 \mid n \in \mathbb{N}\}$. The invariant $\mathcal{C}(M)$ is also computed in [87] and arises as the set of dimensions of some finite dimensional von Neumann algebras. In [64], Deprez and Vaes constructed concrete group-measure space II_1 factors M with $\mathcal{C}(M)$ ranging over all sets of natural numbers that are closed under taking divisors and taking lowest common multiples.

All above results provide II_1 factors M for which $\mathcal{C}(M)$ is a subset of the natural numbers. However, combining recent work on tensor categories [103] and our Theorem 4.A, we prove the following theorem.

Theorem 4.B. *There exists a II_1 factor M such that*

$$\mathcal{C}(M) = \Big\{1, \frac{5+\sqrt{13}}{2}, 12+3\sqrt{13}, 4+\sqrt{13}, \frac{11+3\sqrt{13}}{2}, \frac{13+3\sqrt{13}}{2}, \frac{19+5\sqrt{13}}{2}, \frac{7+\sqrt{13}}{2}\Big\} \, .$$

In [215], [87] and [64] only categories with at most countably many isomorphism classes of irreducible objects were obtained as bimodule categories of II_1 factors. We give examples of II_1 factors M such that $\mathrm{Bimod}(M)$ can be calculated and has uncountably many pairwise non isomorphic irreducible objects. For example, if G is a countable, discrete group, we prove the existence of a II_1 factor M such that $\mathrm{Bimod}(M) \simeq \mathrm{URep}_{\mathrm{fin}}(G)$. Here, $\mathrm{URep}_{\mathrm{fin}}(G)$ denotes the category of finite dimensional representations of G.

Theorem 4.C. *Let $\mathcal{C}$ denote one of the following compact tensor C^*-categories. Either $\mathcal{C} = \mathrm{URep}_{\mathrm{fin}}(G)$ for a countable discrete group, or $\mathcal{C} = \mathrm{UCorep}_{\mathrm{fin}}(A)$ for an amenable or a maximally almost periodic discrete Kac algebra A. Then, there is a II_1 factor M such that $\mathrm{Bimod}(M) \simeq \mathcal{C}$.*

Our construction consists of two main steps.

1. Given any quasi-regular, depth 2 inclusion $N \subset Q$ of II_1 factors, such that N and $N' \cap Q$ are hyperfinite, denote by $N \subset Q \subset Q_1$ the Jones basic construction. We construct a II_1 factor M and a fully faithful tensor C*-functor $F : \mathrm{Bimod}(Q \subset Q_1) \to \mathrm{Bimod}(M)$ (see Section 4.2.4 for the bimodule category associated with an inclusion of II_1 factors).

2. Using Ioana, Peterson and Popa's rigidity results for amalgamated free product von Neumann algebras [117], we prove that under suitable assumptions (see Theorem 4.3.1) the functor F is essentially surjective.

The above steps yield a II_1 factor M such that $\mathrm{Bimod}(M) \simeq \mathrm{Bimod}(Q \subset Q_1)$. Using the setting of [117], as in [86, 87, 216], this result is not constructive. We only prove an existence theorem, which involves a Baire category argument (see Theorem 4.2.19). More precisely, we prove the following Theorem 4.D and Theorems 4.A and 4.C are obtained as corollaries.

Theorem 4.D. *Let $N \subset Q$ be a quasi-regular and depth 2 inclusion of II_1 factors. Assume that N and $N' \cap Q$ are hyperfinite and denote by $N \subset Q \subset Q_1$ the basic construction. Then, there exist uncountably many pairwise non-stably isomorphic II_1 factors (M_i) such that for all i we have $\mathrm{Bimod}(M_i) \simeq \mathrm{Bimod}(Q \subset Q_1)$ as tensor C^*-categories.*

4.2 Preliminaries and notations

In this chapter, von Neumann algebras are assumed to act on a separable Hilbert space. A von Neumann algebra (M, τ) endowed with a faithful normal tracial state τ is called a tracial von Neumann algebra. We define $L^2(M)$ as the GNS Hilbert space with respect to τ.

Whenever M is a von Neumann algebra, we write $M^n = M_n(\mathbb{C}) \otimes M$ and $M^\infty = B(\ell^2(\mathbb{N})) \overline{\otimes} M$. Whenever $\mathcal{H}$ is a Hilbert space, we also denote $\mathcal{H}^\infty = \ell^2(\mathbb{N}) \otimes \mathcal{H}$.

If $B \subset M$ is a tracial inclusion of von Neumann algebras, then we denote by E_B the trace preserving conditional expectation of M onto B. Also if $pB^np \subset pM^np$ is an amplification of $B \subset M$, we still denote by E_B the trace preserving conditional expectation onto pB^np.

4.2.1 Finite index bimodules

Let M, N be tracial von Neumann algebras. An M-N-bimodule ${}_M\mathcal{H}_N$ is a Hilbert space $\mathcal{H}$ equipped with a normal representation of M and a normal anti-representation of N that commute. Bimodules over von Neumann algebras were studied in [53, V.Appendix B] and [160].

Let $\mathcal{H}$ be an M-N-bimodule. There exists a projection $p \in N^\infty$ such that

$$\mathcal{H}_N \cong \big(p\,L^2(N)^\infty\big)_N \ ,$$

and this projection p is uniquely defined up to equivalence of projections in N^∞. There also exists a $*$-homomorphism $\psi : M \to pN^\infty p$ such that ${}_M\mathcal{H}_N$ is isomorphic with the M-N-bimodule $\mathcal{H}(\psi)$ defined as Hilbert space $pL^2(N)^\infty$ and endowed with actions given by

$$a \cdot \xi = \psi(a)\xi \quad \text{and} \quad \xi \cdot b = \xi b \quad \text{and} \quad a \in M \quad , \quad b \in N \quad , \quad \xi \in pL^2(N)^\infty \ .$$

Furthermore, if $\psi : M \to pN^\infty p$ and $\eta : M \to qN^\infty q$, then ${}_M\mathcal{H}(\psi)_N \cong {}_M\mathcal{H}(\eta)_N$ if and only if there exists $u \in N^\infty$ satisfying $uu^* = p$, $u^*u = q$ and $\psi(a) = u\eta(a)u^*$ for all $a \in M$.

Note that M-N-bimodules ${}_M\mathcal{H}_N$ can also be described by means of right actions of $*$-homomorphisms $\psi : N \to pM^\infty p$ as

$${}_M\mathcal{H}_N \cong {}_M\big((\ell^2(\mathbb{N})^* \otimes L^2(M))p\big)_{\psi(N)} \ .$$

Let $\mathcal{H}$ be a right N-module and write $\mathcal{H}_N \cong \big(p\,L^2(N)^\infty\big)_N$, for a projection $p \in N^\infty$. Denote $\dim_{\text{-}N}(\mathcal{H}) = (\mathrm{Tr} \otimes \tau)(p)$. Observe that the number $\dim_{\text{-}N}(\mathcal{H})$ depends on the choice of the trace τ, if N is not a factor.

An M-N-bimodule ${}_M\mathcal{H}_N$ is said to be of *finite Jones index* if $\dim_{M\text{-}}(\mathcal{H}) < +\infty$ and $\dim_{\text{-}N}(\mathcal{H}) < +\infty$. In particular, the *Jones index* of a subfactor $N \subset M$

is defined as $[M : N] = \dim_{-N}(\mathrm{L}^2(M))$, see [125]. Using the above notations, consider a bimodule of the form ${}_M\mathcal{H}(\psi)_N$ with finite Jones index. Then, one may assume that ψ is a finite index inclusion $\psi : M \to pN^n p$.

4.2.2 Popa's intertwining-by-bimodules technique

In [166, Section 2], Popa introduced a very powerful technique to deduce unitary conjugacy of two von Neumann subalgebras A and B of a tracial von Neumann algebra M from their *embedding* $A \prec_M B$, using *interwining bimodules*. When $A, B \subset M$ are Cartan subalgebras of a II_1 factor M, Popa proves [164, Theorem A.1] that $A \prec_M B$ if and only if A and B are actually conjugated by a unitary in M. We also recall the notion of *full embedding* $A \prec_M^f B$ of A into B inside M.

Definition 4.2.1. Let M be a tracial von Neumann and $A, B \subset M^n$ be possibly non-unital subalgebras. We write

- $A \prec_M B$ if $1_A \mathrm{L}^2(M^n) 1_B$ contains a non-zero A-B-subbimodule $\mathcal{K}$ that satisfies $\dim_{-B}(\mathcal{K}) < \infty$.
- $A \prec_M^f B$ if $Ap \prec_M B$ for every non-zero projection $p \in 1_A M^n 1_A \cap A'$.

We will use the following characterization of embedding of subalgebras. It can be found in [166, Theorem 2.1 and Corollary 2.3] (see also Appendix F in [44]).

Theorem 4.2.2 (See [166])**.** *Let M be a tracial von Neumann algebra and $A, B \subset M^n$ possibly non-unital subalgebras. The following are equivalent.*

- *$A \prec_M B$,*
- *there exist $m \in \mathbb{N}$, a $*$-homomorphism $\psi : A \to pB^m p$ and a non-zero partial isometry $v \in 1_A\big(\mathrm{M}_{1,m}(\mathbb{C}) \otimes M^n\big)p$ satisfying $av = v\psi(a)$ for all $a \in A$,*
- *there is no sequence of unitaries $u_k \in \mathcal{U}(A)$ such that $\|\mathrm{E}_B(x u_k y)\|_2 \to 0$ for all $x, y \in M^n$.*

Note that the entries of v as in in the previous theorem span an A-B-bimodule $\mathcal{K} \subset \mathrm{L}^2(M^n)$ such that $\dim_{-B}(\mathcal{K}) < \infty$.

We will make use of Theorem 4.2.4 due to Vaes, [215, Theorem 3.11]. We first recall the notion of essentially finite index inclusions of II_1 factors (see

[215, Proposition A.2]) and embedding of von Neumann subalgebras inside a bimodule.

Let $N \subset M$ be an inclusion of tracial von Neumann algebras. We say that $N \subset M$ has *essentially finite index* if there exists a sequence of projections $p_n \in N' \cap M$ such that p_n tends to 1 strongly and $Np_n \subset p_n M p_n$ has finite Jones index for all n.

Definition 4.2.3. Let M, N be tracial von Neumann algebras and $A \subset M$, $B \subset N$ von Neumann subalgebras. Let ${}_M\mathcal{H}_N$ be an M-N-bimodule. We write

- $A \prec_{\mathcal{H}} B$ if $\mathcal{H}$ contains a non-zero A-B-subbimodule $\mathcal{K} \subset \mathcal{H}$ with $\dim_{\text{-}B}(\mathcal{K}) < \infty$.
- $A \prec^f_{\mathcal{H}} B$ if every non-zero A-N-subbimodule $\mathcal{K} \subset \mathcal{H}$ satisfies $A \prec_{\mathcal{K}} B$.

Denote by τ the trace on M. Let $\mathcal{H}$ be an M-N-bimodule. Using notations from Section 4.2.1, write $\mathcal{H} \cong \mathcal{H}(\psi)$ where ψ is a $*$-homomorphism $\psi : M \to pN^\infty p$ and p a projection in N^∞. Suppose that $\dim_{\text{-}N}(\mathcal{H}) < +\infty$, i.e $(\mathrm{Tr} \otimes \tau)(p) < +\infty$. Then, as remarked in [215], one has

- $A \prec_{\mathcal{H}} B$ if and only if $\psi(A) \prec_N B$,
- $A \prec^f_{\mathcal{H}} B$ if and only if $\psi(A) \prec^f_N B$.

Theorem 4.2.4 ([215, Theorem 3.11])**.** *Let N, M be tracial von Neumann algebras, with trace τ. Let $A \subset M$, $B \subset N$ be von Neumann subalgebras. Assume the following.*

- *Every A-A-subbimodule $\mathcal{K} \subset \mathrm{L}^2(M)$ satisfying $\dim_{\text{-}A}(\mathcal{K}) < +\infty$ is included in $\mathrm{L}^2(A)$.*
- *Every B-B-subbimodule $\mathcal{K} \subset \mathrm{L}^2(N)$ satisfying $\dim_{\text{-}B}(\mathcal{K}) < +\infty$ is included in $\mathrm{L}^2(B)$.*

Suppose that ${}_M\mathcal{H}_N$ is a finite index M-N-bimodule such that $A \prec^f_{\mathcal{H}} B$ and $A \succ^f_{\mathcal{H}} B$. Then there exists a projection $p \in B^\infty$ satisfying $(\mathrm{Tr} \otimes \tau)(p) < +\infty$ and a $*$-homomorphism $\psi : M \to pN^\infty p$ such that

$$ {}_M\mathcal{H}_N \cong {}_M\mathcal{H}(\psi)_N \qquad , \qquad \psi(A) \subset pB^\infty $$

and this last inclusion has essentially finite index.

4.2.3 Amalgamated free products of tracial von Neumann algebras

Throughout this section we consider von Neumann algebras M_0, M_1 endowed with faithful normal tracial states τ_0, τ_1. Let N be a common von Neumann subalgebra of M_0 and M_1 such that the traces τ_0 and τ_1 coincide on N. We denote $M = M_0 *_N M_1$ the amalgamated free product of M_0 and M_1 over N with respect to the trace preserving conditional expectations (see [162] and [226]). Recall that M is endowed with a conditional expectation $E : M \to N$ and the pair (M, E) is unique up to E-preserving isomorphism. The von Neumann algebra $M_0 *_N M_1$ is equipped with a trace defined by $\tau = \tau_0 \circ E = \tau_1 \circ E$.

Rigid subalgebras

Kazhdan's property (T) was generalized to tracial von Neumann algebras by Connes and Jones in [56] and is defined as follows. A II_1 factor M has property (T) if and only if there exists $\epsilon > 0$ and a finite subset $F \subset M$ such that every M-M-bimodule that has a unit vector ξ satisfying $\|x\xi - \xi x\| \leqslant \epsilon$, for all $x \in F$, actually has a non-zero vector ξ_0 satisfying $x\xi_0 = \xi_0 x$, for all $x \in M$.

Note that a group Γ such that every non-trivial conjugacy class is infinite (ICC group) has property (T) in the sense of Kazhdan if and only if the II_1 factor $\mathrm{L}(\Gamma)$ has property (T) in the sense of Connes and Jones.

Popa defined a notion of relative property (T) for inclusions of tracial von Neumann algebras, see [164, Definition 4.2]. Such an inclusion is also called *rigid*. In particular, if N is a II_1 factor having property (T), then any inclusion $N \subset M$ in a finite von Neumann algebra M is rigid.

We will make use of the following characterization of relative property (T).

Theorem 4.2.5 (See [164] and [157]). *An inclusion $N \subset M$ of tracial von Neumann algebras is rigid if and only if every sequence (ψ_n) of trace preserving, completely positive, unital maps $\psi_n : M \to M$ converging to the identity pointwise in $\| \ \|_2$, converges uniformly in $\| \ \|_2$ on the unital ball $(N)_1$ of N.*

We recall Ioana, Peterson and Popa's Theorem 5.1 from [117] which controls the position of rigid subalgebras of amalgamated free product von Neumann algebras. We choose to work with matrices over amalgamated free products, which is not a more general situation, since $(M_0 *_N M_1)^n$ can be identified with $M_0^n *_{N^n} M_1^n$.

Theorem 4.2.6 (See [117, Theorem 4.3]). *Let $M = M_0 *_N M_1$. Let $p \in M^n$ be a projection and $Q \subset pM^n p$ a rigid inclusion. Then there exists $i \in \{0,1\}$ such that $Q \prec_M M_i$.*

Control of quasi-normalizers

Let M be a tracial von Neumann algebra and $N \subset M$ a von Neumann subalgebra. The *quasi-normalizer* of N inside M, denoted $\mathrm{QN}_M(N)$, is defined as the set of elements $a \in M$ for which there exist $a_1, \ldots, a_n, b_1, \ldots, b_m \in M$ such that

$$Na \subset \sum_{i=1}^{n} a_i N, \qquad , \qquad aN \subset \sum_{i=1}^{m} N b_i \ .$$

The inclusion $N \subset M$ is called *quasi-regular* if $\mathrm{QN}_M(N)'' = M$. One also defines the *group of normalizing unitaries* $\mathcal{N}_N(M)$ of $N \subset M$ as the set of unitaries $u \in M$ satisfying $uNu^* = N$. The *normalizer* of N in M is $\mathcal{N}_M(N)''$. Note that $N' \cap M \subset \mathcal{N}_M(N)'' \subset \mathrm{QN}_M(N)''$.

Theorem 4.2.7 (See [117, Theorem 1.1]). *Let $M = M_0 *_N M_1$. Let $p \in M_0^n$ be a projection and $Q \subset pM_0^n p$ a von Neumann subalgebra satisfying $Q \nprec_{M_0} N$. Whenever $\mathcal{K} \subset p(\mathbb{C}^n \otimes \mathrm{L}^2(M))$ is a Q-M_0-subbimodule with $\dim_{\text{-}M_0}(\mathcal{K}) < +\infty$, we have $\mathcal{K} \subset p(\mathbb{C}^n \otimes \mathrm{L}^2(M_0))$. In particular, the quasi-normalizer of Q inside $pM^n p$ is contained in $pM_0^n p$.*

4.2.4 Tensor C*-categories, fusion algebras and bimodule categories of II$_1$ factors

We briefly recall some definitions for tensor C*-categories and refer to [134, 189] for more information and precise statements. A tensor C*-category is a C*-category with a *monoidal structure*, such that all structure maps are unitary. A tensor C*-category is called *regular* if it has subobjects and direct sums and the unit object is strongly irreducible. A regular tensor C*-category is called *compact* if every object has a conjugate. A compact tensor C*-category is *finite* if it has only finitely many isomorphism classes of simple objects.

Convention. Throughout this chapter we assume without loss of generality that all tensor categories involved are strict.

Fusion algebras

A fusion algebra $\mathcal{A}$ is a free $\mathbb{N}$-module $\mathbb{N}[\mathcal{G}]$ equipped with

- an associative and distributive product operation, and a multiplicative unit element $e \in \mathcal{G}$,
- an additive, anti-multiplicative, involutive map $x \mapsto \overline{x}$, called *conjugation*,

satisfying Frobenius reciprocity as follows. For $x, y, z \in \mathcal{G}$, define $m(x, y; z) \in \mathbb{N}$ such that

$$xy = \sum_z m(x, y; z)z \ .$$

Then, one has $m(x, y; z) = m(\overline{x}, z; y) = m(z, \overline{y}; x)$ for all $x, y, z \in \mathcal{G}$.

The base $\mathcal{G}$ of the fusion algebra $\mathcal{A}$, also called the *irreducible elements* of $\mathcal{A}$, consists of the non-zero elements of $\mathcal{A}$ that cannot be expressed as the sum of two non-zero elements.

We have the following examples of fusion algebras.

- Given a countable group Γ, one gets the associated fusion algebra $\mathcal{A} = \mathbb{N}[\Gamma]$.
- Let G be a locally compact group and define the fusion algebra $\mathcal{A}$ of $\mathrm{URep}_{\mathrm{fin}}(G)$ as the set of equivalence classes of finite dimensional unitary representations of G. The direct sum and tensor product of representations in $\mathrm{URep}_{\mathrm{fin}}(G)$ yield a fusion algebra structure on $\mathcal{A}$.
- More generally, the isomorphism classes of objects in a compact tensor C*-category form a fusion algebra. Note that there exist non-equivalent tensor C*-categories having isomorphic fusion algebras.

We are mainly interested in tensor C*-categories and fusion algebras coming from bimodules over II_1 factors. Let us recall some definitions and refer to [36] for background material and results on bimodules and fusion algebras, in particular in relation with subfactors.

The bimodule category of a II_1 factor

Let M, N, P be II_1 factors. We denote by $\mathcal{H} \otimes_N \mathcal{K}$ the Connes tensor product of the M-N-bimodule $\mathcal{H}$ and the N-P-bimodule $\mathcal{K}$ and refer to [53, V.Appendix B] for details. Note that $\mathcal{H}(\rho) \otimes_N \mathcal{H}(\psi) \cong \mathcal{H}((\mathrm{id} \otimes \psi)\rho)$.

We recall now the following useful lemma from [87] concerning Connes tensor product versus product in a given module. The inclusion of II_1 factors $N \subset M$ considered in [87] is assumed to be irreducible ($N' \cap M = \mathbb{C}1$). Instead, we assume that $N \subset M$ is quasi-regular. We give a proof for the convenience of the reader.

Lemma 4.2.8 ([87, Lemma 2.2]). *Let $\widetilde{N} \subset N \subset M$ be an inclusion of II_1 factors and let P be a II_1 factor. Assume that $N \subset M$ is quasi-regular and $\widetilde{N} \subset N$ has finite index. Let ${}_M\mathcal{H}_P$ be an M-P-bimodule. Suppose that $\mathcal{L} \subset \mathcal{H}$ is a closed $\widetilde{N}$-P-subbimodule. Suppose that $\mathcal{K} \subset L^2(M)$ is an N-$\widetilde{N}$-subbimodule of finite index. Denote by $\mathcal{K} \cdot \mathcal{L}$ the closure of $(\mathcal{K} \cap M)\mathcal{L}$ inside $\mathcal{H}$. Then*

- *$\mathcal{K} \cdot \mathcal{L}$ is an N-P-bimodule isomorphic to a subbimodule of $\mathcal{K} \otimes_{\widetilde{N}} \mathcal{L}$.*
- *If $\mathcal{K} \cdot \mathcal{L}$ is non-zero and $\mathcal{K} \otimes_{\widetilde{N}} \mathcal{L}$ is irreducible then, $\mathcal{K} \cdot \mathcal{L}$ and $\mathcal{K} \otimes_{\widetilde{N}} \mathcal{L}$ are isomorphic N-P-bimodules.*

Whenever ${}_P\mathcal{H}_M$ is a P-M-bimodule with closed P-$\widetilde{N}$-subbimodule $\mathcal{L}$ and $\mathcal{K} \subset L^2(M)$ an $\widetilde{N}$-N-subbimodule, we define $\mathcal{L} \cdot \mathcal{K}$ as the closure of $\mathcal{L}(\mathcal{K} \cap M)$ inside $\mathcal{H}$ and, by symmetry, we find that $\mathcal{L} \cdot \mathcal{K}$ is isomorphic with a P-N-subbimodule of $\mathcal{L} \otimes_N \mathcal{K}$.

Proof. Let $\mathcal{H}, \mathcal{K}$ and $\mathcal{L}$ be as in the statement of the lemma. Note that $\mathcal{K} \cap M$ is dense in $\mathcal{K}$, since $N \subset M$ is quasi-regular and $\widetilde{N} \subset N$ has finite index. Moreover, all vectors in $\mathcal{K} \cap M$ are $\widetilde{N}$-bounded. So, there exists a finite index inclusion $\psi : N \to p\widetilde{N}^n p$ and an N-$\widetilde{N}$-bimodular isomorphism $T : \mathcal{H}(\psi) = p(\mathbb{C}^n \otimes L^2(\widetilde{N})) \to \mathcal{K}$ such that $T(p(e_i \otimes 1)) \in \mathcal{K} \cap M$ for all i. We have $\mathcal{K} \otimes_{\widetilde{N}} \mathcal{L} \cong p(\mathbb{C}^n \otimes \mathcal{L})$, hence we can define an N-P-bimodular map $S : p(\mathbb{C}^n \otimes \mathcal{L}) \to \mathcal{K} \cdot \mathcal{L}$ by $S(p(e_i \otimes \xi)) = T(p(e_i \otimes 1)) \cdot \xi$. The range of T is dense in $\mathcal{K} \cdot \mathcal{L}$. After taking the polar decomposition of T we get a coisometry $\mathcal{K} \otimes_N \mathcal{L} \to \mathcal{K} \cdot \mathcal{L}$. □

The *contragredient* of an M-N-bimodule ${}_M\mathcal{H}_N$ is the N-M-bimodule defined on the conjugate Hilbert space $\overline{\mathcal{H}}$ with bimodule actions given by $a \cdot \overline{\xi} = \overline{(\xi a^*)}$ and $\overline{\xi} \cdot b = \overline{(b^* \xi)}$.

The Connes tensor product and contragredience induce a compact tensor C*-category structure on the category of finite index M-M bimodules, where morphisms are given by bimodular maps.

Definition 4.2.9. Let M be a II_1 factor. We define $\mathrm{Bimod}(M)$ to be the tensor C*-category of finite index M-M-bimodules and $\mathrm{Falg}(M)$ the associated fusion algebra.

We recall the notion of pairs of conjugates in strict tensor C*-categories.

Definition 4.2.10 (See [134])**.** Let x be an object in a strict tensor C*-category $\mathcal{C}$. A conjugate for x is an object $\overline{x}$ in $\mathcal{C}$ and morphisms $R : 1_{\mathcal{C}} \to \overline{x} \otimes x$, $\overline{R} : 1_{\mathcal{C}} \to x \otimes \overline{x}$ such that

$$(\overline{R}^* \otimes \mathrm{id}_x) \circ (\mathrm{id}_x \otimes R) = \mathrm{id}_x \quad \text{and} \quad (R^* \otimes \mathrm{id}_{\overline{x}}) \circ (\mathrm{id}_{\overline{x}} \otimes \overline{R}) = \mathrm{id}_{\overline{x}}\,.$$

In the following theorem, pairs of conjugates are used to characterize finite index bimodules among all bimodules over a II_1 factor (see [134] and also [84, Theorem 5.32]).

Theorem 4.2.11. *Let M be a II_1 factor and let ${}_M\mathcal{H}_M$ an M-M-bimodule. Then ${}_M\mathcal{H}_M$ has finite index if and only if ${}_M\mathcal{H}_M$ has a conjugate in the tensor C*-category of all M-M-bimodules.*

Tensor C*-categories arising from subfactors

Let M be a II_1 factor and $N \subset M$ a subfactor. Write e_N for the projection $\mathrm{L}^2(M) \to \mathrm{L}^2(N)$. The von Neumann algebra $\langle M, e_N\rangle \subset \mathrm{B}(\mathrm{L}^2(M))$ generated by M and e_N, called the *Jones basic construction*, was introduced in [125] and is denoted M_1. Note that $\mathrm{L}^2(M_1)$ is an M-M-bimodule and it is of finite Jones index whenever $[M : N] < +\infty$. We will frequently use the fact that $\dim(N' \cap M) < +\infty$ if $[M : N] < +\infty$.

Definition 4.2.12. Let $N \subset M$ be an inclusion of type II_1 factors. We define $\mathrm{Bimod}(N \subset M)$ to be the tensor C*-subcategory of $\mathrm{Bimod}(N)$ generated by all finite index N-N-bimodules that appear in $\mathrm{L}^2(M)$. We denote by $\mathrm{Falg}(N \subset M)$ the associated fusion subalgebra of $\mathrm{Bimod}(N \subset M)$.

We give the following definition of depth 2, as in [80].

Definition 4.2.13. Let $N \subset Q$ be an inclusion of II_1 factors. Let $N \subset Q \subset Q_1 \subset Q_2 \subset \cdots$ be the Jones tower. Then $N \subset Q$ has depth 2 if $N' \cap Q \subset N' \cap Q_1 \subset N' \cap Q_2$ is a basic construction.

Identify $N' \cap Q_2$ with the space $\mathrm{B}_{N\text{-}Q}(\mathrm{L}^2(Q_1, \mathrm{Tr}))$ of bounded N-Q-bimodular maps. Denote by $\mathrm{Hom}_{N\text{-}Q}(\mathrm{L}^2(Q), \mathrm{L}^2(Q_1))$ the Hilbert space completion of N-Q-bimodular maps from $\mathrm{L}^2(Q, \tau)$ to $\mathrm{L}^2(Q_1, \mathrm{Tr})$ with respect to the scalar product $\langle T, S\rangle = \tau(S^*T)$. We recall the following special case of [80, Theorem 3.10].

Theorem 4.2.14 (See [80, Theorem 3.10])**.** *The inclusion $N \subset Q$ of II_1 factors has depth 2 if and only if the natural action of $N' \cap Q_2$ on* $\mathrm{Hom}_{N\text{-}Q}(\mathrm{L}^2(Q), \mathrm{L}^2(Q_1))$ *is faithful.*

As a consequence, we obtain the following characterization of depth 2 inclusions that we use in this chapter.

Corollary 4.2.15. *Let N, Q be II_1 factors. Then, the inclusion $N \subset Q$ has depth 2 if and only if ${}_N\mathrm{L}^2(Q_1)_Q$ is isomorphic to an N-Q-subbimodule of ${}_N\mathrm{L}^2(Q)^\infty{}_Q$.*

Proof. Let $N \subset Q$ be a depth 2 inclusion of II_1 factors. Let $p \in N' \cap Q_2$ be the projection onto the orthogonal complement of the maximal N-Q-subbimodule of $\mathrm{L}^2(Q_1)$ which is contained in ${}_N\mathrm{L}^2(Q)^\infty{}_Q$. Then, p acts trivially on $\mathrm{Hom}_{N\text{-}Q}(\mathrm{L}^2(Q), \mathrm{L}^2(Q_1))$. Therefore, $p = 0$ by Theorem 4.2.14.

Assume that ${}_N\mathrm{L}^2(Q_1)_Q$ is isomorphic to a subbimodule of ${}_N\mathrm{L}^2(Q)^\infty{}_Q$. Let $p \in N' \cap Q_2$ be a non-zero projection. Then $p\mathrm{L}^2(Q_1)$ is a non-zero N-Q-bimodule, so there is a non-trivial N-Q-bimodular map $T : \mathrm{L}^2(Q) \to p\mathrm{L}^2(Q_1)$. We have $p \cdot T = T \neq 0$, so p acts non-trivially on $\mathrm{Hom}_{N\text{-}Q}(\mathrm{L}^2(Q), \mathrm{L}^2(Q_1))$. We have proven that $N' \cap Q_2$ acts faithfully on $\mathrm{Hom}_{N\text{-}Q}(\mathrm{L}^2(Q), \mathrm{L}^2(Q_1))$. We conclude using again Theorem 4.2.14. □

The fusion algebra of almost-normalizing bimodules

Let $N \subset M$ be a regular inclusion, i.e. $\mathcal{N}_M(N)'' = M$. For any element $u \in \mathcal{N}_M(N)$ the N-N-bimodule $u\mathrm{L}^2(N)$ has finite index and lies in $\mathrm{L}^2(M)$. Such bimodules are generalized by the notion of bimodules almost-normalizing the inclusion $N \subset M$, which was introduced by Vaes in [216]. This notion was adapted to more general irreducible, quasi-regular inclusions of II_1 factors $N \subset M$ in [87]. We recall the definition.

Definition 4.2.16. Let $N \subset M$ be an irreducible and quasi-regular inclusion of type II_1 factors. A finite index N-N-bimodule is said to almost-normalize the inclusion $N \subset M$, inside $\mathrm{Falg}(N)$, if it arises as a finite index N-N-subbimodule of a finite index M-M-bimodule. We denote by $\mathrm{AFalg}(N \subset M)$ the fusion algebra generated by N-N-bimodules almost-normalizing the inclusion $N \subset M$.

Let N be a II_1 factor and Γ a countable group acting outerly on N. Write $M = N \rtimes \Gamma$ and assume that the inclusion $N \subset N \rtimes \Gamma$ is rigid. It is proven in [216, Lemma 4.1] that the fusion algebra $\mathrm{AFalg}(N \subset N \rtimes \Gamma)$ is a countable fusion subalgebra of $\mathrm{Falg}(N)$. The next lemma is a straightforward adaptation of [216, Lemma 4.1].

Lemma 4.2.17. *Let $N \subset M$ be a rigid, irreducible and quasi-regular inclusion of type II_1 factors. Then, the fusion algebra* $\mathrm{AFalg}(N \subset M)$ *is a countable fusion subalgebra of* $\mathrm{Falg}(N)$.

Freeness of fusion algebras

The notion of freeness of fusion algebras was introduced in [37, Section 1.2], in the study of free composition of subfactors. We recall the definition.

Definition 4.2.18 ([37, Section 1.2])**.** Let $\mathcal{A}$ be a fusion algebra and $\mathcal{A}_0, \mathcal{A}_1 \subset \mathcal{A}$ fusion subalgebras. We say that $\mathcal{A}_0$ and $\mathcal{A}_1$ are *free inside* $\mathcal{A}$ if every alternating product of irreducibles in $\mathcal{A}_i \backslash \{e\}$, remains irreducible and different from e.

Let M be a II_1 factor and ${}_M\mathcal{K}_M$ a finite index M-M-bimodule. Whenever $\alpha \in \mathrm{Aut}(M)$, we define the conjugation of $\mathcal{K}$ by α as the bimodule $\mathcal{K}^\alpha = \mathcal{H}(\alpha^{-1}) \otimes_M \mathcal{K} \otimes_M \mathcal{H}(\alpha)$. Denote by R the hyperfinite II_1 factor. Vaes proved in [216, Theorem 5.1] that countable fusion subalgebras of $\mathrm{Falg}(R)$ can be made free by conjugating one of them with an automorphism of R (see Theorem 4.2.19 below). Note that the same result has first been proven for countable subgroups of $\mathrm{Out}(R)$ in [117]. In both cases, the key ingredients come from [163].

Theorem 4.2.19 ([216, Theorem 5.1])**.** *Let R be the hyperfinite II_1 factor and $\mathcal{A}_0, \mathcal{A}_1$ two countable fusion subalgebras of* $\mathrm{Falg}(R)$*. Then,*

$$\{\alpha \in \mathrm{Aut}(R) \mid \mathcal{A}_0^\alpha \text{ and } \mathcal{A}_1 \text{ are free}\}$$

is a dense G_δ-subset of $\mathrm{Aut}(R)$*.*

4.3 Proof of Theorem 4.D

We recall the following construction, from [87]. Consider the group $\Gamma = \mathbb{Q}^3 \oplus \mathbb{Q}^3 \rtimes \mathrm{SL}_3(\mathbb{Q})$, defined by the action $A \cdot (x,y) = (Ax, (A^t)^{-1}y)$ of $\mathrm{SL}_3(\mathbb{Q})$ on $\mathbb{Q}^3 \oplus \mathbb{Q}^3$. Take $\alpha \in \mathbb{R} - \mathbb{Q}$, define $\Omega_\alpha \in \mathrm{Z}^2(\mathbb{Q}^3 \oplus \mathbb{Q}^3, S^1)$ such that

$$\Omega_\alpha\big((x,y),(x',y')\big) = \exp\big(2\pi i\alpha(\langle x,y'\rangle - \langle y,x'\rangle)\big)\,, \qquad \text{for all } (x,y),(x',y') \in \mathbb{Q}^3 \oplus \mathbb{Q}^3\,,$$

and extend Ω_α to an S^1-valued 2-cocycle on Γ by $\mathrm{SL}_3(\mathbb{Q})$-invariance. Write $\Lambda = \mathbb{Z}^3 \oplus \mathbb{Z}^3$. Then, by [87, Lemma 3.3] and [87, Example 3.4], the inclusions $N \subset N_0 \subset P$ given by

$$N = \mathrm{L}_{\Omega_\alpha}(\Lambda), \quad N_0 = \mathrm{L}_{\Omega_\alpha}(\mathbb{Z}^3 \oplus \mathbb{Z}^3 \rtimes \mathrm{SL}_3(\mathbb{Z})), \quad P = \mathrm{L}_{\Omega_\alpha}(\Gamma)$$

satisfy the following properties.

($\mathcal{P}_1$) $N \subset P$ is irreducible and quasi-regular,

($\mathcal{P}_2$) $N_0 \subset P$ is quasi-regular,

($\mathcal{P}_3$) N_0 has property (T).

Note that ($\mathcal{P}_1$) follows from the fact that the inclusion $\Lambda \subset \Gamma$ is almost-normal, meaning the commensurator $\mathrm{Comm}_\Gamma(\Lambda)$ defined as

$$\mathrm{Comm}_\Gamma(\Lambda) := \{g \in \Gamma \mid g\Lambda g^{-1} \cap \Lambda \text{ has finite index in } g\Lambda g^{-1} \text{and in } \Lambda\}$$

is the whole of Γ. We know that the group $\mathrm{SL}_3(\mathbb{Q})$ does not have any non-trivial finite dimensional unitary representations (see [229]). The smallest normal subgroup of Γ containing $\mathrm{SL}_3(\mathbb{Q})$ is Γ itself. This gives the following property.

($\mathcal{P}_4$) The group Γ has no non-trivial finite dimensional unitary representations.

We will also need the following additional property, proven in [87, Example 3.4].

($\mathcal{P}_5$) The inclusion $\mathrm{L}_{\Omega_\alpha}(\Lambda_0) \subset \mathrm{L}_{\Omega_\alpha}(\Gamma)$ is irreducible, for every finite index subgroup $\Lambda_0 < \Lambda$.

Theorem 4.3.1. *Let Q be a II$_1$ factor such that $N \subset Q$. Let $B = N' \cap Q$ and assume that*

- *$N \subset Q$ is a quasi-regular and depth 2 inclusion,*
- *B is hyperfinite,*
- *there is no non-trivial $*$-homomorphism from N_0 to any amplification of Q,*
- *the fusion algebras* $\mathrm{AFalg}(N \subset P)$ *and* $\mathrm{Falg}(N \subset Q)$ *defined in Section 4.2.4 are free inside* $\mathrm{Falg}(N)$*.*

Then, for $M = \left(P \overline{\otimes} B\right) *_{N \overline{\otimes} B} Q$, we have that $\mathrm{Bimod}(M) \simeq \mathrm{Bimod}(Q \subset Q_1)$*, as tensor C*-categories, where $N \subset Q \subset Q_1$ is the basic construction.*

Outline of the proof of Theorem 4.D. We first prove Theorem 4.3.1 in two steps. In Section 4.3.1, we construct a fully faithful tensor C*-functor $F : \mathrm{Bimod}(Q \subset Q_1) \to \mathrm{Bimod}(M)$. In Section 4.3.2, we prove that F is essentially surjective, which completes the proof of Theorem 4.3.1. In Section 4.3.3, we give a proof of Theorem 4.D, relying on Theorem 4.3.1.

In the rest of Section 4.3 will always use the notation of Theorem 4.3.1.

4.3.1 A fully faithful functor $F : \mathrm{Bimod}(Q \subset Q_1) \to \mathrm{Bimod}(M)$

Denote by $\mathcal{C}$ the tensor C*-category whose objects are finite index inclusions $\psi : Q \to pQ^\infty p$ with $p \in B^\infty$, $(\mathrm{Tr} \otimes \tau)(p) < \infty$ and $\psi(x) = xp$ for all $x \in N$. The tensor product on $\mathcal{C}$ is given by $\psi_1 \otimes_{\mathcal{C}} \psi_2 = (\mathrm{id} \otimes \psi_2) \circ \psi_1$. Morphisms of $\mathcal{C}$ are given by

$$\mathrm{Hom}_{\mathcal{C}}(\psi_1, \psi_2) = \{T \in qB^\infty p \,|\, \forall x \in Q : T\psi_1(x) = \psi_2(x)T\}\,.$$

Proposition 4.3.2. *The natural inclusion $I : \psi \mapsto \mathcal{H}(\psi)$ of $\mathcal{C}$ into* $\mathrm{Bimod}(Q)$ *defines an equivalence of tensor C*-categories $\mathcal{C} \simeq$* $\mathrm{Bimod}(Q \subset Q_1)$.

Proof. It is easy to check that I is a faithful tensor C*-functor. We prove that I is full and that its essential range is $\mathrm{Bimod}(Q \subset Q_1)$.

We first prove that I is full. Let $T : p\mathrm{L}^2(Q)^\infty \to q\mathrm{L}^2(Q)^\infty$ be a Q-Q-bimodular map between $\mathcal{H}(\psi_1)$ and $\mathcal{H}(\psi_2)$. Then $T \in pQ^\infty q$, since T is right Q-modular. We have $Txp = xqT$ for all $x \in N$, so it follows that $T \in pB^\infty q$. This proves that I is full.

Let us prove that the image of I is contained in $\mathrm{Bimod}(Q \subset Q_1)$. Take a finite index inclusion $\psi : Q \to pQ^\infty p$ with $p \in B^\infty$, $(\mathrm{Tr} \otimes \tau)(p) < \infty$ and $\psi(x) = xp$ for all $x \in N$ and let $\mathcal{H} = \mathcal{H}(\psi)$. We claim that $\mathcal{H}$ is a Q-Q-subbimodule of $\mathrm{L}^2(Q_1)^\infty$. Extend ψ to a map $\mathrm{L}^2(Q) \to \mathrm{L}^2(pQ^\infty p)$ and note that its entries, considered as operators on $\mathrm{L}^2(Q)$, lie in Q_1. Any non-zero column of ψ defines a partial isometry $v \in p(\mathrm{M}_{\infty \times 1}(\mathbb{C}) \overline{\otimes} Q_1)$ satisfying $vx = \psi(x)v$, for all $x \in Q$. Note that $vv^* \in \psi(Q)' \cap pQ_1^\infty p$. If $p \neq vv^*$, then we may apply the previous procedure to the non-zero Q-Q-bimodule $(p - vv^*) \cdot \mathcal{H} \cong \mathcal{H}(\psi(\cdot)(p - vv^*))$. Take a maximal family of non-zero partial isometries v_i inside $p(\mathrm{M}_{\infty \times 1}(\mathbb{C}) \overline{\otimes} Q_1)$ satisfying $\psi(x)v_i = v_i x$ for all $x \in Q$ and such that $v_i v_i^*$ are pairwise distinct orthogonal projections. Consider the projection $r = p - \sum v_i v_i^*$. If $r \neq 0$ then we can apply the previous procedure to the non-zero bimodule $r \cdot \mathcal{H}$. As above, we get a non-zero partial isometry $w \in r(\mathrm{M}_{\infty \times 1}(\mathbb{C}) \overline{\otimes} Q_1)$ such that $\psi(x)w = wx$, for all $x \in Q$. Then, ww^* is orthogonal to all of the $v_i v_i^*$, which contradicts maximality of the family. So, $p = \sum v_i v_i^*$. Putting all these partial isometries in a row, we get an element $u \in p(Q_1)^\infty$ such that $ux = \psi(x)u$, for all $x \in Q$ and $uu^* = \sum v_i v_i^* = p$. This proves our claim.

We now prove that every bimodule $\mathcal{H}$ of $\mathrm{Bimod}(Q \subset Q_1)$ is contained in the essential range of I. Assume that $\mathcal{H}$ arises as a Q-Q-subbimodule of $\mathrm{L}^2(Q_1)^{\otimes_Q k}$, for some $k \in \mathbb{N}$. We prove that $\mathcal{H}$ is a subbimodule of $\mathrm{L}^2(Q_1)^\infty$. By Corollary 4.2.15, we have that $\mathcal{H}$ is isomorphic, as N-Q-bimodule, to a subbimodule of $\mathrm{L}^2(Q)^\infty$. Writing $\mathcal{H} \cong \mathcal{H}(\psi)$, for some finite index inclusion $\psi : Q \to qQ^n q$, we find a non-zero N-central vector $v \in q(\mathrm{M}_{n \times 1}(\mathbb{C}) \otimes \mathrm{L}^2(Q))$.

Taking polar decomposition, we may assume that $v \in q(\mathrm{M}_{n\times 1}(\mathbb{C}) \otimes Q)$ is a partial isometry satisfying $\psi(x)v = vx$, for all $x \in N$. As a consequence, we have $v^*v \in B$. As in the previous paragraph, take a maximal family of non-zero partial isometries v_i inside $q(\mathrm{M}_{n\times 1}(\mathbb{C})\otimes Q)$ satisfying $\psi(x)v_i = v_i x$ for all $x \in N$ and $q = \sum v_i v_i^*$. Putting all partial isometries v_i in one row, we get an element $u \in q(\mathrm{M}_{n\times\infty}(\mathbb{C})\overline{\otimes}Q)$ such that $\psi(x)u = ux$ for all $x \in N$ and $uu^* = \sum v_i v_i^* = q$. Define $p = u^*u$ and note that $p \in B^\infty$. Conjugating with u^* from the beginning yields a map $\psi : Q \to pQ^\infty p$ such that $\psi(x) = px$, for all $x \in N$ and still satisfying $\mathcal{H} \cong \mathcal{H}(\psi)$. □

Take a finite index inclusion $\psi : Q \to pQ^\infty p$ in $\mathcal{C}$. Then, we have $p \in B^\infty$. Denote by $\iota : P\overline{\otimes}B \to p(P\overline{\otimes}B)^\infty p$ the inclusion map given by $x \mapsto xp$ on P and by the restriction $\psi|_B$ on B. Since ψ preserves N, it also preserves $B = N' \cap Q$ and we obtain a map $\iota * \psi : M \to pM^\infty p$. If $T \in \mathrm{Hom}_{\mathcal{C}}(\psi_1, \psi_2)$, then $T \in qB^\infty p$. So, T defines an M-M-modular map from $\mathcal{H}(\iota * \psi_1)$ to $\mathcal{H}(\iota * \psi_2)$. We conclude that the map

$$F_0 : \mathcal{C} \to \mathrm{Bimod}(M) : \psi \mapsto \mathcal{H}(\iota * \psi)$$

is a functor.

Proposition 4.3.3. *F_0 is a fully faithful tensor C*-functor.*

Proof. It is clear that F_0 is faithful. We first prove that F_0 is full. Take $T \in \mathrm{Hom}_{M\text{-}M}(\mathcal{H}(\iota * \psi_1), \mathcal{H}(\iota * \psi_2))$. Then $T : p\mathrm{L}^2(M)^\infty \to q\mathrm{L}^2(M)^\infty$ is right M-modular, hence $T \in pM^\infty q$. Since $Txp = xqT$ for all $x \in P$, we have $T \in pB^\infty q$. So T is in the image of F_0. The functor $*$ on both $\mathrm{Bimod}(Q \subset Q_1)$ and $\mathrm{Bimod}(M)$ is given by $T \mapsto T^*$, so F is a C*-functor. Since $\mathcal{H}(\psi_1) \otimes_M \mathcal{H}(\psi_2) \cong \mathcal{H}((\mathrm{id} \otimes \psi_2) \circ \psi_1)$, it follows immediately that F_0 is a tensor C*-functor. □

Now let $G : \mathrm{Bimod}(Q \subset Q_1) \to \mathcal{C}$ be an inverse functor for the inclusion $I : \mathcal{C} \to \mathrm{Bimod}(Q \subset Q_1)$. We define the fully faithful tensor C*-functor $F = F_0 \circ G : \mathrm{Bimod}(Q \subset Q_1) \to \mathrm{Bimod}(M)$.

4.3.2 Proof of Theorem 4.3.1: essential surjectivity of F

We give a series of preliminary lemmas before proving that the functor F constructed in the previous section is essentially surjective.

Lemma 4.3.4. *Let ${}_M\mathcal{H}_M$ be a finite index M-M-bimodule and ${}_{P\overline{\otimes}B}\mathcal{K}_{P\overline{\otimes}B} \subset {}_{P\overline{\otimes}B}\mathcal{H}_{P\overline{\otimes}B}$ be a finite index $P\overline{\otimes}B$-$P\overline{\otimes}B$-subbimodule. Then $\mathcal{K}$ contains a non-zero N-N-subbimodule $\mathcal{L}$ such that* $\dim_{\text{-}N}(\mathcal{L}) < +\infty$.

Proof. Let $\psi : M \to pM^n p$ and $\varphi : P\overline{\otimes}B \to q(P\overline{\otimes}B)^k q$ be finite index inclusions such that ${}_M\mathcal{H}_M \cong {}_M\mathcal{H}(\psi)_M$ and ${}_{P\overline{\otimes}B}\mathcal{K}_{P\overline{\otimes}B} \cong {}_{P\overline{\otimes}B}\mathcal{H}(\varphi)_{P\overline{\otimes}B}$. Take a non-zero partial isometry $v_0 \in p(\mathrm{M}_{n,k}(\mathbb{C}) \otimes M)q$ such that $\psi(x)v_0 = v_0\varphi(x)$, for all $x \in P\overline{\otimes}B$. We have $v_0^* v_0 \in \varphi(P \otimes B)' \cap qM^k q$, so the support projection $\operatorname{supp} \mathrm{E}_{P\overline{\otimes}B}(v_0^* v_0)$ lies in $\varphi(P\overline{\otimes}B)' \cap q(P\overline{\otimes}B)^k q$. Moreover $v_0 \mathrm{E}_{P\overline{\otimes}B}(v_0^* v_0) = v_0$. So we can assume that $q = \operatorname{supp} \mathrm{E}_{P\overline{\otimes}B}(v_0^* v_0)$.

We claim that $\varphi(N_0) \prec_{P\overline{\otimes}B} P$. Recall that B is hyperfinite, by assumption. Let $\bigcup_n A_n$ be the dense union of an increasing sequence of finite dimensional von Neumann subalgebras A_n of B. Since $P \otimes 1 \subset P \otimes A_n$ is a finite index inclusion for every n, it suffices to show that $\varphi(N_0) \prec_{P\overline{\otimes}B} P \otimes A_n$ for some n. Denote by E_n the trace-preserving conditional expectation of B onto A_n. Then the sequence of unital completely positive maps $\mathrm{id} \otimes \mathrm{E}_n$ on $(P\overline{\otimes}B)^k$, still denoted by E_n, converges pointwise in $\| \ \|_2$ to the identity. Since N_0 has property (T) (see $(\mathcal{P}_3)$), Theorem 4.2.5 shows that (E_n) converges uniformly in $\| \ \|_2$ on $(\varphi(N_0))_1$. Take $n \in \mathbb{N}$ such that $\|\mathrm{E}_n(x) - x\|_2 < 1/2$ for all $x \in \varphi(N_0)$. Assume that $\varphi(N_0) \nprec_{P\overline{\otimes}B} P \otimes A_n$. By Theorem 5.6.1, there is a sequence of unitaries $u_k \in \mathcal{U}(\varphi(N_0))$ such that for all $x, y \in q(P\overline{\otimes}B)^k q$, we have $\|\mathrm{E}_n(x u_k y)\|_2 \to 0$ for $k \to \infty$. In particular,

$$1 = \|u_k\|_2 < 1/2 + \|\mathrm{E}_n(u_k)\|_2 \to 1/2\,,$$

which is a contradiction. We have proved our claim.

This yields a $*$-homomorphism $\pi : N_0 \to rP^l r$ and a non-zero partial isometry $v_1 \in q(\mathrm{M}_{k,l}(\mathbb{C}) \otimes P\overline{\otimes}B)r$ such that $\varphi(x)v_1 = v_1\pi(x)$, for all $x \in N_0$. Similarly to the first paragraph, we can assume that $r = \operatorname{supp} E_P(v_1^* v_1)$. Note that $\mathrm{E}_{P\overline{\otimes}B}(v_0^* v_0) = q$. So $v = v_0 v_1 \in p(\mathrm{M}_{n,l} \otimes M)r$ is a non-zero partial isometry. Moreover, we have $v\pi(x) = \psi(x)v$, for all $x \in N_0$.

We claim that $\pi(N) \prec_P N$. We first prove that it suffices to show that $\pi(N) \prec_{P\overline{\otimes}B} N\overline{\otimes}B$. Indeed, if this the case, we get a $*$-homomorphism $\theta : N \to t(N\overline{\otimes}B)^j t$ and a non-zero partial isometry $u \in r(\mathrm{M}_{l,j}(\mathbb{C}) \otimes P\overline{\otimes}B)t$ such that $\pi(x)u = u\theta(x)$, for all $x \in N$. Denote by u_i the i-th column of u. Then the closed linear span of $\{u_i N\overline{\otimes}B \mid i = 1, \ldots, j\}$ defines a non-zero $\pi(N)$-$N\overline{\otimes}B$-subbimodule of $r(\mathbb{C}^l \otimes \mathrm{L}^2(P\overline{\otimes}B))$ with finite right dimension. Using the N-N-modular projection onto $r(\mathbb{C}^l \otimes \mathrm{L}^2(P))$ and the action of B, we find a non-zero $\pi(N)$-N-bimodule inside $r(\mathbb{C}^l \otimes \mathrm{L}^2(P))$ which is finitely generated as a right N-module.

Assume now that $\pi(N) \nprec_{P\overline{\otimes}B} N\overline{\otimes}B$. Then, Theorem 4.2.7 implies that the quasi-normalizer of $\pi(N)$ in $rM^l r$ sits inside $r(P\overline{\otimes}B)^l r$. As a consequence, $v^* v \in \pi(N)' \cap rM^l r \subset r(P\overline{\otimes}B)^l r$. Since the inclusion $N \subset M$ is quasi-regular, we have that

$$v^* \psi(M) v \subset r(P\overline{\otimes}B)^l r\,. \tag{4.1}$$

Denote by A the von Neumann algebra generated by $\psi(M)$ and vv^*. Then $\psi(M) \subset A \subset pM^n p$ and $A \subset pM^n p$ has finite index. Using (4.1), we get that $v^*Av \subset v^*v(P\overline{\otimes}B)^l v^*v \subset v^*M^n v$, from which we deduce that $P\overline{\otimes}B \subset M$ has finite index. We get a contradiction. Indeed, M being an amalgamated free product, we can find in $\mathrm{L}^2(M)$ infinitely many pairwise orthogonal $P\overline{\otimes}B$-$P\overline{\otimes}B$-bimodules by means of alternating powers of $\mathrm{L}^2(P\overline{\otimes}B) \ominus \mathrm{L}^2(N\overline{\otimes}B)$ and $\mathrm{L}^2(Q) \ominus \mathrm{L}^2(N\overline{\otimes}B)$.

The previous claim yields a $*$-homomorphism $\rho : N \to sN^m s$ and a non-zero partial isometry $w \in r(\mathrm{M}_{l,m}(\mathbb{C}) \otimes P)s$ such that $\pi(x)w = w\rho(x)$ for all $x \in N$. Denote by w_i the i-th column of w. Define $\mathcal{L}$ as the closed linear span of $\{v_1 w_i N \mid i = 1, \dots, m\}$. Then, $\mathcal{L}$ is a non-zero, since $\mathrm{E}_P(w^* v_1^* v_1 w) = w^* \mathrm{E}_P(v_1^* v_1) w$ and $r = \operatorname{supp} \mathrm{E}_P(v_1^* v_1)$. So $\mathcal{L}$ is a non-zero $\varphi(N)$-N-subbimodule of $\mathcal{K}$ with finite right dimension. □

Lemma 4.3.5. *Let $\mathcal{K}$ be a finite index $P\overline{\otimes}B$-$P\overline{\otimes}B$-subbimodule of a finite index M-M-bimodule $\mathcal{H}$ and let ${}_N\mathcal{L}_N \subset {}_N\mathcal{K}_N$ be an irreducible finite index N-N-subbimodule. Then ${}_N\mathcal{L}_N$ is isomorphic to a subbimodule of ${}_N\mathrm{L}^2(P)_N$.*

Proof. Assume, by contradiction, that $\mathcal{L}$ is not contained in ${}_N\mathrm{L}^2(P)_N$. Take some non-trivial finite index irreducible N-N-bimodule $\mathcal{L}^Q$ in $\mathrm{L}^2(Q)$ and some non-trivial finite index irreducible N-N-bimodule $\mathcal{L}^P$ in $\mathrm{L}^2(P)$ both with right dimension greater than or equal to 1. Denote by $\mathcal{X}_0$ the $\|\ \|_2$-closure of $\mathcal{L} \cdot M$. Lemma 4.2.8 implies that $\mathcal{X}_0$ is a non-zero N-M-bimodule which is isomorphic to a subbimodule of $\mathcal{H}$ and lies in $\mathcal{L} \otimes_N \mathrm{L}^2 M$. Define the N-M-bimodules

$$\mathcal{X}_n = (\mathcal{L}^P \otimes_N \mathcal{L}^Q)^{\otimes n} \otimes_N \mathcal{X}_0 \,.$$

Note that $\mathcal{L} \in \mathrm{AFalg}(N \subset P)$. By assumption, the fusion algebras $\mathrm{Falg}(N \subset Q)$ and $\mathrm{AFalg}(N \subset P)$ are free inside $\mathrm{Falg}(N)$. Therefore, as in [87], the $\mathcal{X}_n$ follow pairwise disjoint as N-N-bimodules and hence pairwise disjoint as N-M-bimodules.

Decompose $\mathcal{X}_0 \subset \mathcal{H}$ is a direct sum of irreducible N-N-bimodules $\mathcal{Y}_i$. Write $(\mathcal{L}^Q)^0 = \mathcal{L}^Q \cap Q$ and $(\mathcal{L}^P)^0 = \mathcal{L}^P \cap P$. Then, $(\mathcal{L}^P)^0 \cdot (\mathcal{L}^Q)^0 \cdots (\mathcal{L}^Q)^0 \cdot \mathcal{Y}_i$ is non-zero. If not, we had

$$M \cdot \mathcal{Y}_i \cdot M = M \cdot (\mathcal{L}^P)^0 \cdot (\mathcal{L}^Q)^0 \cdots (\mathcal{L}^Q)^0 \cdot N \cdot \mathcal{Y}_i \cdot M = 0 \,,$$

contradicting the fact that M is a factor. As above, the freeness assumption implies that $(\mathcal{L}^P \otimes_N \mathcal{L}^Q)^{\otimes n} \otimes_N \mathcal{Y}_i$ is irreducible. Then, by Lemma 4.2.8, we have that $(\mathcal{L}^P \otimes_N \mathcal{L}^Q)^{\otimes n} \otimes_N \mathcal{Y}_i$ sits inside $\mathcal{H}$ as the $\| \cdot \|_2$-closure of $(\mathcal{L}^P)^0 \cdot (\mathcal{L}^Q)^0 \cdots (\mathcal{L}^Q)^0 \cdot \mathcal{Y}_i$. We have proven that $\mathcal{H}$ contains a copy of each $\mathcal{X}_n$.

Note that $\dim_{\text{-}M}(\mathcal{X}_n) > \dim_{\text{-}M}(\mathcal{X}_0)$. As a consequence, $\mathcal{H}$, as a right M-module, has infinite dimension, which is a contradiction. □

Lemma 4.3.6. *Let $\mathcal{H}$ be a finite index M-M-bimodule and $\mathcal{K}$ a finite index $P\overline{\otimes}B$-$P\overline{\otimes}B$-subbimodule. Then, $\mathcal{K}$ is isomorphic to a multiple of the trivial P-P bimodule.*

Proof. By Lemma 4.3.4, we have a non-zero N-N-subbimodule $\mathcal{L} \subset \mathcal{K}$ with finite right dimension. Then, for $x, y \in \mathrm{QN}_{P\overline{\otimes}B}(N)$, the closure of $Nx \cdot \mathcal{L} \cdot yN$ is still an N-N-subbimodule of $\mathcal{K}$ with finite right dimension. Since $N \subset P\overline{\otimes}B$ is quasi-regular, the linear span of all N-N-subbimodules of $\mathcal{K}$ with finite right dimension is dense in $\mathcal{K}$. Then, a maximality argument shows that $\mathcal{K}$ can be decomposed as the direct sum N-N-subbimodules with finite right dimension. By symmetry, $\mathcal{K}$ decomposes as the direct sum of N-N-bimodules with finite left dimension. As a consequence, $\mathcal{K}$ may be written as the direct sum of finite index N-N-subbimodules.

Let $\mathcal{L}$ be an irreducible finite index N-N-subbimodule of $\mathcal{K}$. Lemma 4.3.5 shows that $\mathcal{L}$ is contained in $\mathrm{L}^2(P)$. Remember that

$$P = \mathrm{L}_\Omega(\mathbb{Q}^3 \oplus \mathbb{Q}^3 \rtimes \mathrm{SL}_3(\mathbb{Q})) \quad , \quad N = \mathrm{L}_\Omega(\mathbb{Z}^3 \oplus \mathbb{Z}^3) \, .$$

Hence, $\mathcal{L}$ arises as the $\|\cdot\|_2$-closure of Nu_gN for some element $g \in \mathbb{Q}^3 \oplus \mathbb{Q}^3 \rtimes \mathrm{SL}_3(\mathbb{Q})$. By almost-normality (see property $(\mathcal{P}_1)$ and the remarks following it), take a finite index subgroup Λ_0 of $\mathbb{Z}^3 \oplus \mathbb{Z}^3$ such that $\mathrm{Ad}(g)(\Lambda_0) \subset \mathbb{Z}^3 \oplus \mathbb{Z}^3$. Denote by $\mathcal{L}_0$ the closure of $Nu_g\mathrm{L}_\Omega(\Lambda_0)$. Then, $\mathcal{L}_0$ is an irreducible N-$\mathrm{L}_\Omega(\Lambda_0)$-subbimodule of $\mathcal{L}$. Note that

$$\mathcal{L}_0 \underset{\mathrm{L}_\Omega(\Lambda_0)}{\otimes} \overline{\mathrm{L}_\Omega(\Lambda_0)u_g^*N} \;\cong\; \mathrm{L}^2(N) \, .$$

Lemma 4.2.8 implies that $\mathcal{K}$ contains a copy of the trivial N-N-bimodule $\mathrm{L}^2(N)$, realized as the $\|\cdot\|_2$-closure of $\mathcal{L}_0u_g^*N$.

Write $\mathcal{K} \cong \mathcal{H}(\psi)$ for some finite index inclusion $\psi : P\overline{\otimes}B \to q(P\overline{\otimes}B)^\infty q$, where $(\mathrm{Tr} \otimes \tau)(q) < \infty$. By the above paragraph, we can take a trivial N-N-bimodule inside $\mathcal{K}$. Then, there is an N-central vector $v \in q\mathrm{L}^2(P\overline{\otimes}B)^\infty$. Taking polar decomposition, we may assume that v is a partial isometry in $q(\mathrm{M}_{\infty\times 1}(\mathbb{C})\overline{\otimes}P\overline{\otimes}B)$ satisfying $\psi(x)v = vx$, for all $x \in N$. Note that $vv^* \in \psi(N)' \cap q(P\overline{\otimes}B)^\infty q$. Hence, $(q - vv^*) \cdot \mathcal{K}$ defines a N-$P\overline{\otimes}B$-subbimodule of $\mathcal{K}$ and we may apply the previous procedure. As in the proof of Proposition 4.3.2, a maximality argument yields a family of partial isometries $v_i \in q(\mathrm{M}_{\infty\times 1}(\mathbb{C})\overline{\otimes}P\overline{\otimes}B)$ satisfying $\psi(x)v_i = v_ix$, for all $x \in N$ and such that $\sum v_iv_i^* = q$. Putting these partial isometries in a row, we obtain an element $w \in q(P\overline{\otimes}B)^\infty$ satisfying $ww^* = \sum v_iv_i^* = q$. By irreducibility of $N \subset P$ (see $(\mathcal{P}_1)$), we have a projection $p = w^*w \in (N' \cap P\overline{\otimes}B)^\infty = B^\infty$. Conjugating ψ with w^* from the beginning, we obtain a finite index inclusion $\psi : P\overline{\otimes}B \to p(P\overline{\otimes}B)^\infty p$, where $p \in B^\infty$ such that $(\mathrm{Tr} \otimes \tau)(p) < \infty$ and $\psi(x) = xp$ for all $x \in N$ and still satisfying $\mathcal{K} \cong \mathcal{H}(\psi)$.

Let $g \in \Gamma = \mathbb{Q}^3 \oplus \mathbb{Q}^3 \rtimes \mathrm{SL}_3(\mathbb{Q})$ and $\Lambda_0 < \mathbb{Z}^3 \oplus \mathbb{Z}^3$ be finite index subgroup such that $\mathrm{Ad}(g^{-1})(\Lambda_0) \subset \Lambda$. Denote by $\alpha_{g^{-1}}$ the $*$-homomorphism $\mathrm{L}_\Omega(\Lambda_0) \to \mathrm{L}_\Omega(\Lambda)$ induced by $\mathrm{Ad}(g^{-1})$. For $x \in \mathrm{L}_\Omega(\Lambda_0)$ we have

$$\psi(u_g)u_g^* x = \psi(u_g)\alpha_{g^{-1}}(x)u_g^* = \psi(u_g)\psi(\alpha_{g^{-1}}(x))u_g^* = x\psi(u_g)u_g^* .$$

By $(\mathcal{P}_5)$, we have that $\mathrm{L}_\Omega(\Lambda_0) \subset P$ is irreducible. As a consequence, $v_g = \psi(u_g)u_g^* \in \mathcal{U}(pB^\infty p)$. Note that $\psi(B) \subset \psi(N)' \cap p(P\overline{\otimes}B)^\infty p = pB^\infty p$ and $v_g \in \psi(B)' \cap pB^\infty p$.

We prove that the inclusion $\psi(B) \subset pB^\infty p$ has finite index using Theorem 4.2.11. Consider the conjugate bimodule $\overline{\mathcal{K}}$ of ${}_{P\overline{\otimes}B}\mathcal{K}_{P\overline{\otimes}B}$. As proven above, we may write $\overline{\mathcal{K}} \cong \mathcal{H}(\overline{\psi})$, where $\overline{\psi} : P\overline{\otimes}B \to q(P\overline{\otimes}B)^\infty q$ is a finite index inclusion satisfying $\overline{\psi}(x) = xq$, for all $x \in N$ and $q \in B^\infty$ is a projection such that $(\mathrm{Tr} \otimes \tau)(q) < \infty$. Note that $\mathcal{K} \otimes_{P\overline{\otimes}B} \overline{\mathcal{K}} \cong \mathcal{H}((\overline{\psi} \otimes \mathrm{id}) \circ \psi)$. Hence there is a conjugate map $R : \mathrm{L}^2(P\overline{\otimes}B) \to \mathcal{H}((\overline{\psi} \otimes \mathrm{id}) \circ \psi)$. Considering R as an element of $(p \otimes q)(\mathrm{M}_{\infty,1}(\mathbb{C}) \otimes P\overline{\otimes}B)$ we have

$$R \in (p \otimes q)(\mathrm{M}_{\infty,1}(\mathbb{C}) \otimes P\overline{\otimes}B) \cap N' = (p \otimes q)(\mathrm{M}_{\infty,1}(\mathbb{C})\overline{\otimes}B) .$$

Define $\psi_B : B \to pB^\infty p$ and $\overline{\psi}_B : B \to qB^\infty q$ as the restrictions of ψ and $\overline{\psi}$ to B and $S : \mathrm{L}^2(B) \to \mathcal{H}((\overline{\psi}_B \otimes \mathrm{id}) \circ \psi_B)$ as the restriction of R. Similarly, we find an intertwiner $\overline{S} : \mathrm{L}^2(B) \to \mathcal{H}((\psi_B \otimes \mathrm{id}) \circ \overline{\psi_B})$, giving a pair of conjugate morphisms for $\mathcal{H}(\psi_B)$. Then the same argument as in [134, Lemma 3.2] implies that $\psi(B)' \cap pB^\infty p$ is of finite type I.

It follows that $g \mapsto v_g \in \psi(B)' \cap pB^\infty p$ is a direct integral of finite dimensional unitary representations of Γ and hence trivial, since Γ has no non-trivial finite dimensional unitary representations (see $(\mathcal{P}_4)$). We conclude that $\psi(u_g) = u_g p$ and that ${}_P\mathcal{K}_P$ is a multiple of the trivial P-P-bimodule. ☐

Lemma 4.3.7. *Let $\psi : P\overline{\otimes}B \to p(P\overline{\otimes}B)^n p$ be a finite index inclusion such that*

$${}_{\psi(P \otimes 1)}\big(p(\mathbb{C}^n \otimes \mathrm{L}^2(P\overline{\otimes}B))\big)_{P \otimes 1}$$

is a multiple of the trivial P-P-bimodule. Then there exists a non-zero partial isometry $u \in \mathrm{M}_{n,\infty}(\mathbb{C})\overline{\otimes}P\overline{\otimes}B$ such that $uu^* = p$, $q = u^*u \in B^\infty$ and $u^*\psi(x)u = qx$ for all $x \in P$, where we consider $P \subset P^\infty$ diagonally.

Proof. Consider the P-P-bimodule $\mathcal{H}$ given by

$${}_P\mathcal{H}_P = {}_{\psi(P \otimes 1)}\big(p(\mathbb{C}^n \otimes \mathrm{L}^2(P\overline{\otimes}B))\big)_{P \otimes 1} .$$

Since $\mathcal{H}$ is a multiple of the trivial P-P-bimodule, there exists a non-zero vector $v \in p\big(\mathbb{C}^n \otimes \mathrm{L}^2(P\overline{\otimes}B)\big)$ such that $\psi(x)v = vx$ for all $x \in P$. Taking its polar decomposition, we may assume that v is a non-zero partial isometry in $p(\mathbb{C}^n \otimes P\overline{\otimes}B)$. As in the proof of Proposition 4.3.2, a maximality argument provides a family of non-zero partial isometries (v_i), inside $p(\mathbb{C}^n \otimes P\overline{\otimes}B)$, satisfying $\psi(x)v_i = v_i x$ for all $x \in P$ and such that $p = \sum v_i v_i^*$. Putting all v_i in one row, we get $u \in \mathrm{M}_{n,\infty}(\mathbb{C})\overline{\otimes}P\overline{\otimes}B$. Then $\psi(x)u = ux$, for all $x \in P$. We also have that $uu^* = \sum v_i v_i^* = p$ and $u^*u \in (1 \otimes P \otimes 1)' \cap (P\overline{\otimes}B)^\infty = B^\infty$. Thus, u is the required partial isometry. □

Proof of Theorem 4.3.1. Let ${}_M\mathcal{H}_M$ be a finite index irreducible M-M-bimodule. We prove that $\mathcal{H}$ is isomorphic to a bimodule in the range of the functor $F : \mathrm{Bimod}(Q \subset Q_1) \to \mathrm{Bimod}(M)$, constructed in Section 4.3.1. We do this in two steps.

Step 1. There exists a projection $p \in (P\overline{\otimes}B)^\infty$ with $(\mathrm{Tr}\otimes\tau)(p) < +\infty$ and $*$-homomorphism $\psi : M \to pM^\infty p$ such that

- $\psi(M) \subset pM^\infty p$ has finite index,
- $\psi(P\overline{\otimes}B) \subset p(P\overline{\otimes}B)^\infty p$ and this inclusion has essentially finite index and
- ${}_M\mathcal{H}_M \cong {}_M\mathcal{H}(\psi)_M$.

Proof of Step 1. Let $\psi : M \to pM^n p$ be a finite index inclusion such that ${}_M\mathcal{H}_M \cong {}_M\mathcal{H}(\psi)_M$. By symmetry, Theorem 4.2.4 and the remarks preceding it, we are left with proving the two following statements.

1. $\psi(P\overline{\otimes}B)q \prec_M P\overline{\otimes}B$, for every projection $q \in \psi(P\overline{\otimes}B)' \cap pM^n p$.
2. Whenever $\mathcal{K} \subset \mathrm{L}^2(M)$ is a $(P\overline{\otimes}B)$-$(P\overline{\otimes}B)$-subbimodule satisfying $\dim_{\text{-}P\overline{\otimes}B}(\mathcal{K}) < +\infty$, we have $\mathcal{K} \subset \mathrm{L}^2(P\overline{\otimes}B)$.

By assumption, there is no non-trivial $*$-homomorphism from N_0 to any amplification of Q. It follows that $\psi(N_0) \nprec_M Q$. Hence, Theorem 4.2.6 implies that $\psi(N_0) \prec_M P\overline{\otimes}B$. So there is a $*$-homomorphism $\varphi : N_0 \to q(P\overline{\otimes}B)^m q$ and a non-zero partial isometry $v \in p(\mathrm{M}_{n,m}(\mathbb{C}) \otimes M)q$ such that $\psi(x)v = v\varphi(x)$ for all $x \in N_0$. We have $v^*v \in \varphi(N_0)' \cap qM^m q$. So $v^*v \in q(P\overline{\otimes}B)^m q$ by Theorem 4.2.7. Then,

$$v^*v(\mathrm{QN}_{qM^mq}(\varphi(N_0))'')v^*v \subset q(P\overline{\otimes}B)^m q\,,$$

by Theorem 4.2.7 again. Since $N_0 \subset P$ is quasi-regular (see $(\mathcal{P}_2)$), we also have that

$$v^*\psi(P\overline{\otimes}B)v \subset v^*v(\mathrm{QN}_{qM^mq}(\varphi(N_0))'')v^*v \subset q(P\overline{\otimes}B)^m q\,.$$

Note that all the previous arguments remain true when cutting down ψ with a projection in $\psi(P\overline{\otimes}B)' \cap pM^n p$, so we have proven (i). Theorem 4.2.7 implies (ii) and Step 1 is proven.

Step 2. We may assume that $p \in B^\infty$ and that the $*$-homomorphism ψ satisfies

- $\psi(x) = px$, for all $x \in P$,
- $\psi(B) \subset B^\infty$,
- $\psi(Q) \subset pQ^\infty p$.

Proof of Step 2. By Step 1, the inclusion $\psi(P\overline{\otimes}B) \subset p(P\overline{\otimes}B)^\infty p$ has essentially finite index. Let q be a projection in $\psi(P\overline{\otimes}B)' \cap p(P\overline{\otimes}B)^\infty p$ such that $\mathcal{K} = {}_{\psi(P\overline{\otimes}B)}\big(q\mathrm{L}^2(P\overline{\otimes}B)^\infty\big)_{P\overline{\otimes}B}$ is a finite index $P\overline{\otimes}B$-$P\overline{\otimes}B$-subbimodule of ${}_{P\overline{\otimes}B}\mathcal{H}_{P\overline{\otimes}B}$. Lemma 4.3.6 implies that ${}_{P\,\otimes\,1}\mathcal{K}_{P\,\otimes\,1}$ is isomorphic to a multiple of the trivial P-P-bimodule. Lemma 4.3.7 yields a non-zero partial isometry $u \in q(\mathrm{M}_{\infty,m}(\mathbb{C}) \otimes P\overline{\otimes}B)$ satisfying $u^*\psi(x)u = u^*ux$ for all $x \in P$ and such that $uu^* = q$ and $u^*u \in B^m$. Since $\psi(P\overline{\otimes}B) \subset p(P\overline{\otimes}B)^\infty p$ has essentially finite index, this procedure provides a non-zero partial isometry $u \in (P\overline{\otimes}B)^\infty$ satisfying $u^*\psi(x)u = u^*ux$ for all $x \in P$ with $uu^* = p$ and $u^*u \in B^\infty$. Conjugating ψ with u^* from the beginning, we may assume that $p \in B^\infty$ and $\psi(x) = px$ for all $x \in P$.

We have $P' \cap M = B$ and $\psi(x) = px$ for all $x \in P$, with $p \in B^\infty$, therefore $\psi(B) \subset B^\infty$.

Since $p \in (N' \cap Q)^\infty$ and $\psi(x) = px$ for all $x \in P$, the $*$-homomorphism ψ extends to an N-N-bimodular map $v : \mathrm{L}^2(M) \to \mathrm{L}^2(pM^\infty p)$. By freeness of $\mathrm{Falg}(N \subset Q)$ and $\mathrm{Falg}(N \subset P)$ inside $\mathrm{Falg}(N)$, we have that $v(\mathrm{L}^2(Q))$ is an N-N-subbimodule of $\mathrm{L}^2(pQ^\infty p)$. Hence $\psi(Q) \subset pQ^\infty p$, which ends the proof of Step 2. □

4.3.3 Proof of Theorem 4.D

We use the following version of [146, Theorem 0.2] for the proof of Theorem 4.D.

Theorem 4.3.8 (See [146, Theorem 0.2])**.** *Let Γ be a property (T) group and M a separable II$_1$ factor. Let $J \subset \mathrm{H}^2(\Gamma, \mathrm{S}^1)$ be the set of scalar 2-cocycles Ω such that there exists a (not necessarily unital) non-trivial $*$-homomorphism from $\mathrm{L}_\Omega(\Gamma)$ to an amplification of M. Then J is countable.*

Proof of Theorem 4.D. Fix an inclusion of II_1 factors $N \subset Q$ and assume that N is hyperfinite. Suppose also that $N \subset Q$ is quasi-regular and has depth 2. Denote by $N \subset Q \subset Q_1$ the basic construction.

Let $\alpha \in \mathbb{R} - \mathbb{Q}$ and consider the groups Λ, Γ and the scalar 2-cocycle $\Omega_\alpha \in Z^2(\Gamma, S^1)$ defined in at the beginning of Section 4.3. Since the group $\mathbb{Z}^3 \oplus \mathbb{Z}^3 \rtimes \mathrm{SL}_3(\mathbb{Z})$ has property (T), Theorem 4.3.8 implies that there are uncountably many $\alpha \in \mathbb{R} - \mathbb{Q}$ such that there is no non-trivial $*$-homomorphism from $N_0 = \mathrm{L}_{\Omega_\alpha}\big(\mathbb{Z}^3 \oplus \mathbb{Z}^3 \rtimes \mathrm{SL}_3(\mathbb{Z})\big)$ to any amplification of Q. Take one such $\alpha \in \mathbb{R} - \mathbb{Q}$. Note that by $(\mathcal{P}_1)$, $(\mathcal{P}_3)$ and Lemma 4.2.17, the fusion algebra $\mathcal{F} = \mathrm{AFalg}\big(\mathrm{L}_{\Omega_\alpha}(\Lambda) \subset \mathrm{L}_{\Omega_\alpha}(\Gamma)\big)$ is countable.

Observe that $\mathrm{L}_{\Omega_\alpha}(\Lambda)$ and N are two copies of the hyperfinite II_1 factor and take an isomorphism $\theta : N \to \mathrm{L}_{\Omega_\alpha}(\Lambda)$. Then, the fusion algebra $\mathcal{F}^\theta$ may be viewed as a fusion subalgebra of $\mathrm{Falg}(N)$. Since $\mathrm{Falg}(N \subset Q)$ is a countable fusion subalgebra of $\mathrm{Falg}(N)$, Theorem 4.2.19 allows us to choose θ such that $\mathcal{F}^\theta$ is free with respect to $\mathrm{Falg}(N \subset Q)$. We identify N and $\mathrm{L}_{\Omega_\alpha}(\Lambda)$ through this isomorphism and all assumptions of Theorem 4.3.1 are satisfied. Write $P_\alpha = \mathrm{L}_{\Omega_\alpha}(\Gamma)$ and write

$$M_\alpha = (P_\alpha \overline{\otimes} B) \underset{N \overline{\otimes} B}{*} Q \quad , \quad \text{where} \quad B = N' \cap Q\,.$$

Using Theorem 4.3.1, we obtain that $\mathrm{Bimod}(M_\alpha) \simeq \mathrm{Bimod}(Q \subset Q_1)$.

We prove that stable isomorphism classes of M_α, $\alpha \in \mathbb{R} - \mathbb{Q}$, are countable. Assume by contradiction that there exists an uncountable subset $J \subset \mathbb{R} - \mathbb{Q}$ such that M_{α_j} are pairwise stably isomorphic, for $j \in J$. We find $k \in J$ and an uncountable subset $I \subset J$ such that M_{α_i} embeds (not necessarily unitally) into M_{α_k}, for all $i \in I$. In particular, $\mathrm{L}_{\Omega_{\alpha_i}}\big(\mathbb{Z}^3 \oplus \mathbb{Z}^3 \rtimes \mathrm{SL}_3(\mathbb{Z})\big)$ embeds into M_{α_k}, for all $i \in I$. Since $\mathbb{Z}^3 \oplus \mathbb{Z}^3 \rtimes \mathrm{SL}_3(\mathbb{Z})$ is a property (T) group and cohomology classes of the cocycles

$$\big(\Omega_\alpha\big)|_{\mathbb{Z}^3 \oplus \mathbb{Z}^3 \rtimes \mathrm{SL}_3(\mathbb{Z})} \quad , \quad \alpha \in \mathbb{R} - \mathbb{Q}$$

are two by two non-equal, this contradicts Theorem 4.3.8. □

4.4 Applications

4.4.1 Examples of categories that arrise as $\mathrm{Bimod}(M)$

In this part, we give examples of categories that arise as $\mathrm{Bimod}(M)$ of some II_1 factor M. Note that the results in [216] and in [87] show that the trivial tensor C*-category and the category of finite dimensional representation of every compact, second countable group can be realized as a category of bimodules.

Finite tensor C*-categories

The following reconstruction theorem for finite tensor C*-categories is well known, but for convenience, we give a short proof. We use Jones' planar algebras [123] and Popa's reconstruction theorem for finite depth standard invariants [161]. See also [35] for a similar statement.

Theorem 4.4.1. *Let $\mathcal{C}$ be a finite tensor C*-category. Then there exists a finite index depth 2 inclusion $Q \subset Q_1$ of hyperfinite II_1 factors such that* $\mathrm{Bimod}(Q \subset Q_1) \simeq \mathcal{C}$.

Proof. We define a depth 2 subfactor planar algebra P, such that the inclusion of hyperfinite II_1 factors $Q \subset Q_1$ associated with it by [161, 123] satisfies $\mathrm{Bimod}(Q \subset Q_1) \simeq \mathcal{C}$. Let $x \in \mathcal{C}$ be the direct sum of representatives for every isomorphism class of irreducible objects in $\mathcal{C}$. Denote by $\overline{x}$ the conjugate object of x. Let

$$P_k := \mathrm{End}(\underbrace{x \otimes \overline{x} \otimes \cdots \otimes x}_{k \text{ factors}}) \ .$$

We prove that $P = \bigcup P_k$ is a subfactor planar algebra. Composition of endomorphisms and the $*$-functor of $\mathcal{C}$ make P a $*$-algebra. The categorical trace of $\mathcal{C}$ defines a positive trace on P. Moreover, the graphical calculus for tensor C*-categories induces an action of the planar operad on P. We have $\dim P_0 = 1$, since $1_{\mathcal{C}}$ is irreducible. Moreover, for all k we have $\dim P_k < \infty$, since $\mathcal{C}$ is finite. Finally, the closed loops represent the number $\dim_{\mathcal{C}} x \neq 0$. So P is a subfactor planar algebra. It has depth 2, since $\dim \mathcal{Z}(P_k)$ is the number of isomorphism classes of irreducible objects of $\mathcal{C}$ for every $k \geqslant 1$ and, in particular, $\dim \mathcal{Z}(P_1) = \dim \mathcal{Z}(P_3)$.

Note that, in the language of [161], finite depth subfactor planar algebras correspond to canonical commuting squares [36, 124]. So, by [161], there is an inclusion $Q \subset Q_1$ of hyperfinite II_1 factors with associated planar algebra $P^{Q \subset Q_1} \cong P$. Then x corresponds to ${}_Q\mathrm{L}^2(Q_1)_{Q_1}$. Let $\mathcal{D} = \mathrm{Bimod}(Q \subset Q_1)$ and denote by $Q \subset Q_1 \subset Q_2$ the basic construction. If p, q are minimal projections in $Q' \cap Q_2$, we canonically identify $\mathrm{Hom}_{Q\text{-}Q}(p\mathrm{L}^2(Q_1), q\mathrm{L}^2(Q_1))$ with $q(Q' \cap Q_2)p$. This defines a C*-functor $F : \mathcal{D} \to \mathcal{C}$ sending $p\mathrm{L}^2(Q_1) \cong p({}_Q\mathrm{L}^2(Q_1) \otimes_{Q_1} \mathrm{L}^2(Q_1)_Q)$ to $p(x \otimes \overline{x})$ and mapping morphisms as given by the identification $P^{Q \subset Q_1} \cong P$. Then F is fully faithful and essentially surjective. We have to prove that F preserves tensor products. Let p,q be projections in $Q' \cap Q_2$. The shift-by-two operator $\mathrm{sh}_2 : P_2 \to P_4$ is defined by adding two strings on the left. By [36], we have $p\mathrm{L}^2(Q_1) \otimes_Q q\mathrm{L}^2(Q_1) \cong p \cdot \mathrm{sh}_2(q)\mathrm{L}^2(Q_2)$ as Q-Q-bimodules. On the other hand, we have $p(x \otimes \overline{x}) \otimes q(x \otimes \overline{x}) \cong (p \otimes q)(x \otimes \overline{x} \otimes x \otimes \overline{x})$ in $\mathcal{C}$. Since under the identification $P_k \cong Q' \cap Q_k$ the shift-by-two operator corresponds to $q \mapsto 1 \otimes q$, we have $\mathcal{C} \simeq \mathcal{D}$ as tensor C*-categories. This completes the proof. □

Proof of Theorem 4.A. Let $\mathcal{C}$ be a finite tensor C*-category. By Theorem 4.D it suffices to show that there is a finite index, depth 2 inclusion $N \subset Q$ of hyperfinite II_1 factors, such that for the basic construction $N \subset Q \subset Q_1$ we have $\mathrm{Bimod}(Q \subset Q_1) \simeq \mathcal{C}$. Indeed, if $N \subset Q$ is of finite index, then it is quasi-regular. By Theorem 4.4.1, there is a finite index depth 2 inclusion $N_{-1} \subset N$ of hyperfinite II_1 factors such that $\mathrm{Bimod}(N_{-1} \subset N) \simeq \mathcal{C}$. Let $N_{-1} \subset N \subset Q \subset Q_1$ be the basic construction. Then $N \subset Q$ is a finite index, depth 2 inclusion and $\mathrm{Bimod}(Q \subset Q_1) \simeq \mathrm{Bimod}(N_{-1} \subset N) \simeq \mathcal{C}$. □

Representation categories

In [87] the categories of finite dimensional representations of compact second countable groups were realized as bimodule categories of a II_1 factor. As already mentioned, this forms a natural class of tensor C*-categories, since they can be abstractly characterized as symmetric tensor C*-categories with at most countably many isomorphism classes of irreducible objects. We realize categories of finite dimensional representations of discrete countable groups and of finite dimensional corepresentations of certain discrete Kac algebras as bimodule categories of a II_1 factor. Neither does this class of categories have an abstract characterization, nor does the finite dimensional corepresentation theory of a discrete Kac algebra describe it completely. However, Corollary 4.4.4 shows that we have interesting applications coming from this class of tensor C*-categories.

For notation concerning quantum groups, we refer the reader to the appendix in Section 4.5.

Definition 4.4.2 (See Section 4.5 and Theorem 4.5 of [198])**.** A discrete Kac algebra A is called maximally almost periodic, if there is a family of finite dimensional corepresentations $U_n \in A \otimes \mathrm{B}(H_{U_n})$ such that $A = \overline{\mathrm{span}}\{(\mathrm{id} \otimes \omega)(U_n) \mid n \in \mathbb{N}, \omega \in \mathrm{B}(H_{U_n})_*\}$

Theorem 4.4.3. *Let A be a discrete Kac algebra admitting a strictly outer action on the hyperfinite II_1 factor. Then there is a II_1 factor M such that* $\mathrm{Bimod}(M) \simeq \mathrm{UCorep}_{\mathrm{fin}}(A^{\mathrm{coop}})$.

Proof. Since A acts strictly outerly on the hyperfinite II_1 factor R, the inclusion $R \subset A \ltimes R \subset \widehat{A}^{\mathrm{coop}} \ltimes A \ltimes R$ is a basic construction by [212, Proposition 2.5 and

Corollary 5.6]. The inclusion $R \subset A \ltimes R$ has depth 2 by [212, Corollary 5.10] and since A is discrete, it is quasi-regular. Moreover, we have $\mathrm{Bimod}(A \ltimes R \subset \hat{A} \ltimes A \ltimes R) \simeq \mathrm{UCorep}_{\mathrm{fin}}(A^{\mathrm{coop}})$ by Theorem 4.5.1. So Theorem 4.D yields a II_1 factor M such that $\mathrm{Bimod}(M) \simeq \mathrm{UCorep}_{\mathrm{fin}}(A^{\mathrm{coop}})$. □

Proof of Theorems 4.C. By Theorem 4.4.3 it suffices to show that every discrete group G and every amenable and every maximally almost periodic Kac algebra A has a strictly outer action on the hyperfinite II_1 factor R.

Let us first consider the case of a discrete group. The non-commutative Bernoulli shift $G \curvearrowright (\mathrm{M}_2(\mathbb{C}), \mathrm{tr})^{\otimes G}$ is well known to be outer. It is clear that $\otimes_{n=1}^{\infty}(\mathrm{M}_2(\mathbb{C}), \mathrm{tr})$ is isomorphic to R.

First note that $(A^{\mathrm{coop}})^{\mathrm{coop}} = A$ for all quantum groups A. By Vaes [213, Theorem 8.2], it suffices to show that every amenable and every maximally almost periodic Kac algebra A there is a faithful corepresentation of A^{coop} in the hyperfinite II_1 factor.

If A is a discrete amenable Kac algebra, then so is A^{coop}. By [213, Proposition 8.1], A^{coop} has a faithful corepresentation into R. If A is a discrete maximally almost periodic Kac algebra, then A^{coop} is also maximally almost periodic, since A has a bounded antipode. There is a countable family of corepresentations U_n of A^{coop} whose coefficients span A densely. Considering $\oplus_n \mathrm{B}(H_{U_n})$ as a von Neumann subalgebra of R, the corepresentation $\boxplus_n U_n$ of A^{coop} is faithful. □

As a corollary of Theorem 4.C, we get the following improvement of [117, Corollary 8.8] and [86]. This is the first example of an explicitly known bimodule category with uncountably many isomorphism classes of irreducible objects.

Corollary 4.4.4. *Let G be a second countable, compact group. Then there is a II_1 factor M such that $\mathrm{Out}(M) \cong G$ and every finite index bimodule of M is of the form $\mathcal{H}(\alpha)$ for some $\alpha \in \mathrm{Aut}(M)$. In particular, the bimodule category of M can be explicitly calculated and has an uncountable number of isomorphism classes of irreducible objects.*

The exact sequence $1 \to \mathrm{Out}(M) \to \mathrm{grp}(M) \to \mathcal{F}(M) \to 1$ shows that the fundamental group of M obtained in Corollary 4.4.4 is trivial. Note, that the factors constructed in [86, 117] also have trivial fundamental group.

Proof. Let G be a second countable, compact group. By [199, Theorem 4.2], $\mathrm{L}(G)$ is maximally almost periodic and its irreducible, finite dimensional corepresentations are one dimensional and indexed by elements of G. Their tensor product is given by multiplication in G. So we can apply Theorem 4.C to the discrete Kac algebra $\mathrm{L}(G)$ in order to obtain M. □

4.4.2 Possible indices of irreducible subfactors

In this section, we investigate the structure of subfactors of the II_1 factor M that we obtained in Theorem 4.A. We write

$$\mathcal{C}(M) = \{\lambda \mid \text{there is an irreducible finite index subfactor of } M \text{ with index } \lambda\} .$$

We use the fact that the lattice of irreducible subfactors of a II_1 factor is actually encoded in its bimodule category. In special cases, indices of irreducible subfactors correspond to Frobenius-Perron dimensions (see [83, Section 8]) of objects in the bimodule category. Using recent work on tensor categories [103] and Theorem 4.A, we give examples of of II_1 factors M such that $\mathcal{C}(M)$ can be computed explicitly and contains irrationals.

Definition 4.4.5 (See [92, 237])**.** Let $\mathcal{C}$ be a compact tensor C*-category with tensor unit $1_{\mathcal{C}}$.

1. An algebra $(\mathcal{A}, m, \eta)$ in $\mathcal{C}$ is an object $\mathcal{A}$ in $\mathcal{C}$ with multiplication and unit maps $m : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$ and $\eta : 1_{\mathcal{C}} \to \mathcal{A}$ such that the following diagrams commute

$$\begin{array}{ccc} \mathcal{A} \otimes \mathcal{A} \otimes \mathcal{A} & \xrightarrow{m \otimes \mathrm{id}} & \mathcal{A} \otimes \mathcal{A} \\ \downarrow{\scriptstyle \mathrm{id} \otimes m} & & \downarrow{\scriptstyle m} \\ \mathcal{A} \otimes \mathcal{A} & \xrightarrow{m} & \mathcal{A} \end{array} \qquad \begin{array}{ccccc} \mathcal{A} \otimes 1_{\mathcal{C}} & \xleftarrow{\cong} & \mathcal{A} & \xrightarrow{\cong} & 1_{\mathcal{C}} \otimes \mathcal{A} \\ \downarrow{\scriptstyle \mathrm{id} \otimes \eta} & & \| & & \downarrow{\scriptstyle \eta \otimes \mathrm{id}} \\ \mathcal{A} \otimes \mathcal{A} & \xrightarrow{m} & \mathcal{A} & \xleftarrow{m} & \mathcal{A} \otimes \mathcal{A} . \end{array}$$

2. A coalgebra $(\mathcal{A}, \Delta, \epsilon)$ in $\mathcal{C}$ is an object $\mathcal{A}$ in $\mathcal{C}$ with comultiplication and counit map $\Delta : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$ and $\epsilon : \mathcal{A} \to 1_{\mathcal{C}}$ such that $(\mathcal{A}, \Delta^*, \epsilon^*)$ is an algebra.

3. A Frobenius algebra $(\mathcal{A}, m, \eta, \Delta, \epsilon)$ in $\mathcal{C}$ is an object $\mathcal{A}$ in $\mathcal{C}$ with maps m, η, Δ, ϵ such that $(\mathcal{A}, m, \eta)$ is an algebra, $\Delta = m^*$, $\epsilon = \eta^*$ and

$$(\mathrm{id} \otimes m) \circ (\Delta \otimes \mathrm{id}) = \Delta \circ m = (m \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \Delta) .$$

4. A Frobenius algebra $(\mathcal{A}, m, \eta, \Delta, \epsilon)$ is special if Δ and η are isometric.

5. A Frobenius algebra $\mathcal{A}$ is irreducible if $\dim(\mathrm{Hom}(1_{\mathcal{C}}, \mathcal{A})) = 1$.

Remark 4.4.6. Note that the notion of a special Frobenius algebra is equivalent to the notion of a Q-system [134].

The following lemma and proposition are probably well known, but since we could not find a reference, we give a short proof for convenience of the reader.

Lemma 4.4.7. *Let $M \subset M_1$ be a finite index inclusion of tracial von Neumann algebras. Then* $\mathrm{L}^2(M_1)$ *is a special Frobenius algebra in* $\mathrm{Bimod}(M)$*. The Frobenius algebra* $\mathrm{L}^2(M_1)$ *is irreducible if and only if $M \subset M_1$ is irreducible.*

Proof. We prove that $\mathrm{L}^2(M_1)$ is an algebra in $\mathrm{Bimod}(M)$ with coisometric multiplication and isometric unit. By [134], this shows that $\mathrm{L}^2(M_1)$ is a special Frobenius algebra. The multiplication on $\mathrm{L}^2(M_1)$ is given by $m(x \otimes_M y) = xy$, for $x, y \in M$. The commutative diagram

$$\begin{array}{ccc} \mathrm{L}^2(M_1) \otimes_M \mathrm{L}^2(M_1) & \xrightarrow{\ m\ } & \mathrm{L}^2(M_1) \\ \downarrow \cong & \nearrow_{e} & \\ \mathrm{L}^2(M_2) & & \end{array}$$

proves that m is well defined and coisometric. Here we denote by $M \subset M_1 \subset M_2$ the basic construction and we denote by e the Jones projection. The unit map of $\mathrm{L}^2(M_1)$ is given by the canonical embedding $\mathrm{L}^2(M) \to \mathrm{L}^2(M_1)$.

The inclusion $M \subset M_1$ is irreducible if and only if ${}_M\mathrm{L}^2(M_1)_{M_1}$ is irreducible if and only if ${}_M\mathrm{L}^2(M_1)_M \cong {}_M\mathrm{L}^2(M_1) \otimes_{M_1} \mathrm{L}^2(M_1)_M$ contains a unique copy of ${}_M\mathrm{L}^2(M)_M$. □

Whenever $\mathcal{H}$ is a finite index M-M-bimodule over a II_1 factor M, we denote by $\mathcal{H}^0$ the set of bounded vectors in $\mathcal{H}$. Recall that $\mathcal{H}^0$ is dense in $\mathcal{H}$.

Proposition 4.4.8. *Let M be a II_1 factor. Then there is a bijection between irreducible special Frobenius algebras in* $\mathrm{Bimod}(M)$ *and irreducible finite index inclusions $M \subset M_1$ of II_1 factors. The bijection is given by*

$$\mathcal{H} \mapsto (M \subset \mathcal{H}^0) \ \textit{and} \ (M \subset M_1) \mapsto {}_M\mathrm{L}^2(M_1)_M \,.$$

Proof. Lemma 4.4.7 shows that $\mathrm{L}^2(M_1)$ is an irreducible special Frobenius algebra for all irreducible finite index inclusions $M \subset M_1$. Let $(\mathcal{H}, m, \epsilon, \Delta, \eta)$ be an irreducible special Frobenius algebra in $\mathrm{Bimod}(M)$. We have to prove that $M \subset \mathcal{H}^0$ is a finite index, irreducible inclusion of von Neumann algebras. Let $M_2 = \mathrm{Hom}_{\text{-}M}(\mathcal{H})$ be the commutant of the right M-action. Then

$$\begin{array}{ccc} \mathcal{H}^0 \otimes \mathcal{H} & \longrightarrow & \mathcal{H} \otimes_M \mathcal{H} \\ & \searrow & \downarrow{\scriptstyle m} \\ & & \mathcal{H} \end{array}$$

yields a map $\phi : \mathcal{H}^0 \to M_2$. By considering the restriction $m : \mathcal{H}^0 \otimes \mathrm{L}^2(M) \to \mathcal{H}$, it is clear that ϕ is injective. Consider the special Frobenius algebra $(\mathrm{L}^2(M_2), m_2, \eta_2, \Delta_2, \epsilon_2)$. We prove that $\phi(\mathcal{H}^0)$ is a Frobenius subalgebra of $\mathrm{L}^2(M_2)$. Indeed, the composition $\mathcal{H}^0 \otimes \mathcal{H}^0 \to \mathcal{H} \otimes_M \mathcal{H} \to \mathcal{H}$ induces a multiplication on $\mathcal{H}^0$, since M-M-bimodular maps send M-bounded vectors to M-bounded vectors. Since m is associative, we have for $\xi, \xi' \in \mathcal{H}^0$

$$\begin{aligned} \phi(m(\xi, \xi')) \cdot \widehat{1_M} &= m \circ (m \otimes \mathrm{id})(\xi, \xi', 1_M) \\ &= m \circ (\mathrm{id} \otimes m)(\xi, \xi', 1_M) = \phi(\xi) \cdot \widehat{\phi(\xi')} = \phi(\widehat{\xi) \cdot \phi}(\xi') \,. \end{aligned}$$

So m is the restriction of m_2. By taking adjoints, we see that Δ is the restriction of Δ_2. Next, note that $m \circ (\eta \otimes \mathrm{id}) = \mathrm{id} = m_2 \circ (\eta_2 \otimes \mathrm{id})$. So $\phi(\eta(x)) \cdot \xi = x\xi$ for all $x \in M \subset \mathrm{L}^2(M)$ and all $\xi \in \mathcal{H}$. So η agrees with η_2. Again, by taking adjoints, ϵ is the restriction of ϵ_2.

Let $R : \mathrm{L}^2(M) \to \overline{\mathcal{H}} \otimes_M \mathcal{H}$ denote the standard conjugate for $\mathcal{H}$ [134]. Frobenius algebras are self-dual via $\Delta \circ \eta$, that is $\Delta \circ \eta : \mathrm{L}^2(M) \to \mathcal{H} \otimes_M \mathcal{H}$ is a conjugate for $\mathcal{H}$. In particular, there is an M-M-bimodular isomorphism $\psi : \mathcal{H} \to \overline{\mathcal{H}}$ such that

$$\begin{array}{ccc} \overline{\mathcal{H}} \otimes_M \mathcal{H} & \xrightarrow{R^*} & \mathrm{L}^2(M) \\ \uparrow{\scriptstyle \psi \otimes \mathrm{id}} & & \uparrow{\scriptstyle \epsilon} \\ \mathcal{H} \otimes_M \mathcal{H} & \xrightarrow{m} & \mathcal{H} \end{array}$$

commutes. Denoting by $R_2 : \mathrm{L}^2(M) \to \overline{\mathrm{L}^2(M_2)} \otimes_M \mathrm{L}^2(M_2)$ the standard conjugate for $\mathrm{L}^2(M_2)$ we have the commuting diagram

$$\begin{array}{ccc} \overline{\mathrm{L}^2(M_2)} \otimes_M \mathrm{L}^2(M_2) & \xrightarrow{R_2^*} & \mathrm{L}^2(M) \\ \Big\uparrow {\scriptstyle (^{-}\circ *)\otimes \mathrm{id}} & & \Big\uparrow {\scriptstyle \epsilon_2} \\ \mathrm{L}^2(M_2) \otimes_M \mathrm{L}^2(M_2) & \xrightarrow{m_2} & \mathrm{L}^2(M_2)\,. \end{array}$$

Note that, by the definition of standard conjugates, R is the composition of R_2 with the orthogonal projection $\overline{\mathrm{L}^2(M_2)} \otimes_M \mathrm{L}^2(M_2) \to \overline{\mathcal{H}} \otimes_M \mathcal{H}$. So ψ is the restriction of $^{-} \circ *$. Now consider the commutative diagram

$$\begin{array}{ccc} \overline{\mathrm{L}^2(M_2)} \otimes_M \mathrm{L}^2(M_2) & \xrightarrow{(R_2^*\otimes\mathrm{id})\circ(\mathrm{id}\otimes\Delta_2)} & \mathrm{L}^2(M) \otimes_M \mathrm{L}^2(M_2) \\ \Big\uparrow {\scriptstyle (^{-}\circ *)\otimes \mathrm{id}} & & {\scriptstyle \epsilon_2\otimes\mathrm{id}} \Big\uparrow \\ \mathrm{L}^2(M_2) \otimes_M \mathrm{L}^2(M_2) & \xrightarrow{(m_2\otimes\mathrm{id})\circ(\mathrm{id}\otimes\Delta_2)} & \mathrm{L}^2(M_2) \otimes_M \mathrm{L}^2(M_2)\,. \end{array}$$

It restricts to the corresponding diagram with $\mathrm{L}^2(M_2)$ replaced by $\mathcal{H}$. Define $\overline{m} = (R^* \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \Delta) : \overline{\mathcal{H}}^0 \otimes \mathcal{H} \to \mathcal{H}$ and $\overline{m_2} = (R_2^* \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \Delta_2)$. Since

$$m_2 = (\epsilon_2 \otimes \mathrm{id}) \circ (m_2 \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \Delta_2)$$

in the Frobenius algebra $\mathrm{L}^2(M_2)$, we have that

$$\begin{array}{ccc} \overline{M_2} \otimes \mathrm{L}^2(M_2) & \xrightarrow{\overline{m_2}} & \mathrm{L}^2(M_2) \\ \Big\uparrow {\scriptstyle (^{-}\circ *)\otimes \mathrm{id}} & & \Big\| \\ M_2 \otimes \mathrm{L}^2(M_2) & \xrightarrow{m_2} & \mathrm{L}^2(M_2) \end{array} \quad \text{and} \quad \begin{array}{ccc} \overline{\mathcal{H}}^0 \otimes \mathcal{H} & \xrightarrow{\overline{m}} & \mathcal{H} \\ \Big\uparrow {\scriptstyle \psi\otimes \mathrm{id}} & & \Big\| \\ \mathcal{H}^0 \otimes \mathcal{H} & \xrightarrow{m} & \mathcal{H}\,. \end{array}$$

commute and the second diagram is a restriction of the first one. Denote by $\overline{\phi} : \overline{\mathcal{H}}^0 \to M_2$ the embedding defined by $\overline{m}$. Then $\overline{\phi}(\overline{x}) = \phi(x)^*$ for $x \in \mathcal{H}^0$ and $\phi(\mathcal{H}^0) = \overline{\phi}(\overline{\mathcal{H}}^0)$. This proves that $\phi(\mathcal{H}^0)$ is closed under taking adjoints.

We already proved that $\phi(\mathcal{H}^0)$ is a $*$-subalgebra of M_2. Since ${}_M\mathcal{H}$ has finite dimension, $\phi(\mathcal{H}^0)$is finitely generated over M. Hence, it is weakly closed in M_2, so it is a von Neumann subalgebra. Finally, ${}_M\mathrm{L}^2(\mathcal{H}^0)_M \cong {}_M\mathcal{H}_M$, so $M \subset \mathcal{H}^0$ is irreducible and has finite index. □

Remark 4.4.9. 1. By uniqueness of multiplicative dimension functions on finite tensor C*-categories, see [45], we have $[M_1 : M]_{\min} = \mathrm{FPdim}({}_M\mathrm{L}^2(M_1)_M)$, where $[M_1 : M]_{\min}$ denotes the minimal index [104, 133] and FPdim denotes the Frobenius Perron dimension [83, Section 8]. So if $M \subset M_1$ is extremal (for example irreducible), then we have $[M_1 : M] = \mathrm{FPdim}({}_M\mathrm{L}^2(M_1)_M)$.

2. By Proposition 4.4.8, irreducible special Frobenius algebras correspond to irreducible inclusions $M \subset M_1$, hence to irreducible subfactors $N \subset M$. In particular, if $\mathrm{Bimod}(M)$ is finite, then $\mathcal{C}(M) = \{\mathrm{FPdim}(\mathcal{H}) \mid \mathcal{H} \text{ irreducible special Frobenius algebra in } \mathrm{Bimod}(M)\}$.

We can prove Theorem 4.B now.

Proof of Theorem 4.B. Denote by $\mathcal{C}$ the Haagerup fusion category [4]. In [103], possible principle graphs of irreducible special Frobenius algebras in $\mathcal{C}$ are classified. Lemma 3.9 in [103] gives a list of possible principle graphs of non-trivial simple algebras in $\mathcal{C}$. Note that the list of indices in Theorem 4.B is the same as the indices of graphs in [103, Lemma 3.9]. We will refer with 1), 2), etc. to the graphs in this lemma. We prove that all the indices of these graphs, are actually realized by some irreducible special Frobenius algebra in $\mathcal{C}$.

Since, by [103, Theorem 3.25], there are three pairwise different categories that are Morita equivalent to $\mathcal{C}$, all the possible principal graphs of minimal simple algebras are are actually realized by some irreducible special Frobenius algebra in $\mathcal{C}$. So the graphs 1) and 3) are realized. Using the notation of [103] for irreducible objects in $\mathcal{C}$, the graphs 4), 6) and 7) are realized by the irreducible special Frobenius algebras $\eta\overline{\eta}$, $\nu\overline{\nu}$ and $\mu\overline{\mu}$. We are left with the graphs 2) and 5). Theorem 3.25 in [103] gives the fusion rules for module categories over $\mathcal{C}$. A short calculation shows that the square of the dimension of the second object in the module category associated with the Haagerup subfactor is the index of the graph 2). This proves that the graph 2) is realized. A similar calculation shows that the second object in the second non-trivial module category over $\mathcal{C}$ gives rise to an irreducible special Frobenius algebra with principal graph given by 5).

So all indices in [103, Lemma 3.9] are actually attained by some irreducible special Frobenius algebra in $\mathcal{C}$. According to Theorem 4.A it is possible to find a II_1 factor M such that $\mathrm{Bimod}(M) \simeq \mathcal{C}$. We conclude using Remark 4.4.9. □

4.5 Categories of unitary corepresentations and bimodule categories of subfactors

In this section, we prove that the category of finite dimensional unitary corepresentations of a discrete Kac algebra A, whose coopposite A^{coop} acts strictly outerly on the hyperfinite II_1 factor R, is realized as the the bimodule category of the inclusion $A \ltimes R \subset \widehat{A} \ltimes A \ltimes R$. For convenience of the reader, we give a short introduction.

4.5.1 Preliminaries on quantum groups

Locally compact quantum groups (see [131])

A locally compact quantum group in the setting of von Neumann algebras is a von Neumann algebra A equipped with a normal $*$-homomorphism $\Delta : A \to A \overline{\otimes} A$ and two normal, semi-finite, faithful weights ϕ, ψ satisfying

- Δ *is comultiplicative:* $(\mathrm{id} \otimes \Delta) \circ \Delta = (\Delta \otimes \mathrm{id}) \circ \Delta$.
- ϕ *is left invariant:* $\phi((\omega \otimes \mathrm{id})(\Delta(x))) = \phi(x)\omega(1_M)$ for all $\omega \in M_*^+$ and all $x \in M^+$.
- ψ *is right invariant:* $\psi((\mathrm{id} \otimes \omega)(\Delta(x))) = \psi(x)\omega(1_M)$ for all $\omega \in M_*^+$ and all $x \in M^+$.

We call Δ the comultiplication of A and ϕ, ψ the left and the right Haar weight of A, respectively. If ϕ and ψ are tracial, then A is called a Kac algebra. If A is of finite type I, then we say that A is discrete. If ϕ and ψ are finite, we say that A is compact.

If Γ is a discrete group, then $\ell^\infty(\Gamma)$ is a discrete Kac algebra with comultiplication given by $\Delta(f)(g, h) = f(gh)$ and the left and right Haar weight both induced by the counting measure on Γ.

For any locally compact quantum group (A, Δ) one can construct a dual locally compact quantum group $(\widehat{A}, \widehat{\Delta})$ and a coopposite locally compact quantum group A^{coop}. They both are represented on the same Hilbert space as A. Hence, it makes sense to write formulas involving elements of A and $\widehat{A}$ at the same time. We have $(A, \Delta) \cong (\widehat{\widehat{A}}, \widehat{\widehat{\Delta}})$ and A is compact if and only if $\widehat{A}$ is discrete.

Corepresentations (see [205])

A unitary corepresentation in H of a locally compact quantum group A is a unitary $U \in A\overline{\otimes}\mathrm{B}(H)$ such that $(\Delta \otimes \mathrm{id})(U) = U_{23}U_{13}$. In what follows, we refer to unitary corepresentations simply as corepresentations. If $U \in \mathrm{B}(H_U)\overline{\otimes}A$ is a corepresentation of A, then we also refer to U_{21} $inA\overline{\otimes}\mathrm{B}(H_U)$ as a corepresentation A. A corepresentation U of A is called finite dimensional if H_U is finite dimensional. The direct sum of two corepresentations U, V of A is denoted by $U \boxplus V \in A\overline{\otimes}(\mathrm{B}(H_U) \oplus \mathrm{B}(H_V)) \cong A\overline{\otimes}\mathrm{B}(H_U) \oplus A\overline{\otimes}\mathrm{B}(H_V)$. The tensor product of two corepresentations U and V is given by $U \boxtimes V = U_{12}V_{13} \in A\overline{\otimes}\mathrm{B}(H_U)\overline{\otimes}\mathrm{B}(H_V)$. An intertwiner between two corepresentations U and V is a bounded linear map $T : H_U \to H_V$ satisfying $(\mathrm{id} \otimes T)U = V(\mathrm{id} \otimes T)$. The space of all intertwiners between U and V is denoted by $\mathrm{Hom}(U, V)$. To every irreducible corepresentation $U \in A\overline{\otimes}\mathrm{B}(H_U)$ of A, one associates its conjugate corepresentation $(* \otimes {}^{-})(U) \in A\overline{\otimes}\mathrm{B}(\overline{H_U})$. Here $\overline{H_U}$ denotes the conjugate Hilbert space of H_U. With this structure, the corepresentations of a locally compact quantum group A become a tensor C*-category UCorep(A). Its maximal compact tensor C*-subcategory is the category of finite dimensional corepresentations $\mathrm{UCorep}_{\mathrm{fin}}(A)$. If A is a compact quantum group, every irreducible corepresentation of A is finite dimensional and every corepresentation is a direct sum of (possibly infinitely many) irreducible corepresentations. Coefficients of tensor products of arbitrary length of its irreducible corepresentations of A span it densely .

Let A denote a compact quantum group. Then we can describe the evaluation of the Haar states on coefficients of corepresentations. In particular, $(\mathrm{id}\otimes\psi)(U) = (\mathrm{id}\otimes\phi)(U) = \delta_{U,\epsilon} \cdot 1$, where $\delta_{U,\epsilon}$ is 1 if U is the trivial corepresentation and 0 otherwise.

If A is discrete, its dual is compact. We can write A as

$$\bigoplus_{U \text{ irr. corep. of } \hat{A}} \mathrm{B}(H_U)\,.$$

For any element $x \in A$ we can characterize $\Delta(x)$ as the unique element in $A\overline{\otimes}A$ that satisfies $\Delta(x)T = Tx$ for all $T \in \mathrm{Hom}(U_1, U_2 \boxtimes U_3)$ and all irreducible corepresentations U_1,U_2 and U_3 of $\hat{A}$. Moreover, we can write any corepresentation $V \in A\overline{\otimes}\mathrm{B}(H_V)$ of A as a direct sum of elements $V_U \in \mathrm{B}(H_U) \otimes \mathrm{B}(H_V)$ where U runs through the irreducible corepresentations of $\hat{A}$. If ϵ denotes the trivial corepresentation, then $V_\epsilon = 1 \otimes 1$. Moreover, $V_{\overline{U}} = ({}^{-} \otimes *)(V)$.

Actions of quantum groups (see [213])

An action of a locally compact quantum group A on a von Neumann algebra N is a normal $*$-homomorphism $\alpha : N \to A \overline{\otimes} N$ such that $(\Delta \otimes \mathrm{id}) \circ \alpha = (\mathrm{id} \otimes \alpha) \circ \alpha$. The crossed product von Neumann algebra of N by α is then the von Neumann algebra $A \ltimes N$ generated by $\hat{A} \otimes 1$ and $\alpha(N)$. We identify N and $\hat{A}$ with subalgebras of $A \ltimes N$. There is a natural action $\hat{\alpha}$ of $\hat{A}$ on $A \ltimes N$, which is uniquely defined by $\hat{\alpha}(a) = \hat{\Delta}(a)$ for $a \in \hat{A}$ and $\hat{\alpha}(x) = 1 \otimes x$ for $x \in N$. This action is called the dual action of α.

If an action $\alpha : N \to A \overline{\otimes} N$ of a locally compact quantum group on a factor satisfies $N' \cap A \ltimes N = \mathbb{C} \cdot 1$, then α is called strictly outer.

Let A be a discrete quantum group that acts via α on a von Neumann algebra N. We denote $A \ltimes N$ by M and as before we identify $\hat{A}$ and N with subalgebras of M. If A is a Kac algebra, N is finite and α preserves a trace τ_N on N, then M is also finite. A faithful normal trace on M is given by

$$(\tau \otimes \mathrm{id})(U(1 \otimes x)) = \delta_{U,\epsilon} \cdot \tau_N(x)\,,$$

for all $x \in N$ and for all irreducible corepresentations $U \in \hat{A} \otimes \mathrm{B}(H_U)$ of $\hat{A}$. For $x \in N$ and $U \in \mathrm{B}(H_U) \otimes \hat{A}$ an irreducible corepresentation of $\hat{A}$, we write $\alpha_U(x)$ for the projection of $\alpha(x)$ onto the direct summand $\mathrm{B}(H_U) \otimes N$ of $A \overline{\otimes} N$. For $x \in N$ we have $U(1 \otimes x)U^* = \alpha_U(x)$.

4.5.2 Corepresentation categories of Kac algebras

Theorem 4.5.1. *Let N be a II_1 factor, A a discrete quantum group and $\alpha : A \to A \overline{\otimes} N$ a strictly outer action. Denote by $M = A \ltimes N$ the crossed product of N by α and write $\hat{A} \ltimes M$ for the crossed product by the dual action. Then* $\mathrm{Bimod}(M \subset \hat{A} \ltimes M) \simeq \mathrm{UCorep}_{\mathrm{fin}}(A^{\mathrm{coop}})$ *as tensor C^*-categories, where* $\mathrm{UCorep}_{\mathrm{fin}}(A^{\mathrm{coop}})$ *denotes the category of finite dimensional corepresentations of A^{coop}.*

Proof. We first construct a fully faithful tensor C*-functor F going from $\mathrm{UCorep}_{\mathrm{fin}}(A^{\mathrm{coop}})$ to $\mathrm{Bimod}(M \subset \hat{A} \ltimes M)$. Then, we show that it is essentially surjective.

Step 1. Let $V \in A \otimes \mathrm{M}_n(\mathbb{C})$ be a finite dimensional corepresentation of A^{coop}, that is $(\Delta \otimes \mathrm{id})(V) = V_{13}V_{23}$. We define a $*$-homomorphism $\psi : M \to \mathrm{M}_n(\mathbb{C}) \otimes M$ such that

$$\psi(x) = 1 \otimes x \;\text{ for all } x \in N \quad \text{and} \quad (\mathrm{id} \otimes \psi)(U) = U_{13}V_{12}\,,$$

where $U \in \mathrm{B}(H_U) \otimes \hat{A} \subset A \overline{\otimes} \hat{A}$ is an irreducible corepresentation of $\hat{A}$.

Proof of Step 1. We first show that ψ defines a $*$-homomorphism. This is obvious on N. In order to prove that ψ is multiplicative on $\hat{A} \subset M$, we have to check for all irreducible corepresentations U_1, U_2, U_3 of $\hat{A}$ and for every intertwiner $T \in \mathrm{Hom}(U_1, U_2 \boxtimes U_3)$ the identity

$$(\mathrm{id} \otimes \psi)(U_2)_{134}(\mathrm{id} \otimes \psi)(U_3)_{234}(T \otimes \mathrm{id}) = (T \otimes \mathrm{id})(\mathrm{id} \otimes \psi)(U_1)$$

holds. We have

$$\begin{aligned}(T \otimes \mathrm{id})(\mathrm{id} \otimes \psi)(U_1) &= (T \otimes \mathrm{id})U_{1,13}V_{U_1,12} \\ &= U_{2,14}U_{3,24}(\Delta \otimes \mathrm{id})(V_{U_1})_{123}(T \otimes \mathrm{id}) \\ &= U_{2,14}U_{3,24}V_{U_2,13}V_{U_3,23}(T \otimes \mathrm{id}) \\ &= U_{2,14}V_{U_2,13}U_{3,24}V_{U_3,23}(T \otimes \mathrm{id}) \\ &= (\mathrm{id} \otimes \psi)(U_2)_{134}(\mathrm{id} \otimes \psi)(U_3)_{234}(T \otimes \mathrm{id})\,.\end{aligned}$$

We prove that ψ is a homomorphism on $\mathrm{alg}(\hat{A}, N) = *\text{-}\mathrm{alg}(\hat{A}, N)$. Using the fact that $U(1 \otimes x) = \alpha_U(x)U$ for all $x \in N$ and all irreducible corepresentations U of $\hat{A}$, it suffices to note that

$$\begin{aligned}(\mathrm{id} \otimes \psi)(U)(1 \otimes 1 \otimes x) &= U_{13}V_{12}(1 \otimes 1 \otimes x) \\ &= U_{13}(1 \otimes 1 \otimes 1 \otimes x)V_{12} \\ &= \alpha_U(x)_{13}U_{13}V_{12} \\ &= \alpha_U(x)_{13}(\mathrm{id} \otimes \psi)(U)\,.\end{aligned}$$

Let us show that ψ is $*$-preserving. We have

$$\begin{aligned}(\mathrm{id} \otimes \psi)((^- \otimes *)(U)) = &= (\mathrm{id} \otimes \psi)(\overline{U}) \\ &= \overline{U}_{13}V_{12} \\ &= \overline{U}_{13}V_{\overline{U},12} \\ &= (^- \otimes *)(U)_{13}(^- \otimes *)(V_U)_{12} \\ &= (^- \otimes * \otimes *)(U_{13}V_{12})\,,\end{aligned}$$

This shows that ψ is a $*$-homomorphism on $*\text{-}\mathrm{alg}(\hat{A}, N)$.

Let us show that ψ is trace preserving on $*$-alg$(\widehat{A}, N)$. Denote by τ the trace on M. For an irreducible corepresentation U of $\widehat{A}$ and $x \in N$ we have

$$(\mathrm{id} \otimes \tau)(U(1 \otimes x)) = \delta_{U,\epsilon} \tau(x) 1_A \,,$$

by the definition of τ. On the other hand we have

$$\begin{aligned}(\mathrm{id} \otimes \mathrm{tr} \otimes \tau)(\mathrm{id} \otimes \psi)(U(1 \otimes x)) &= (\mathrm{id} \otimes \mathrm{tr} \otimes \tau)(U_{13} V_{12}(1 \otimes 1 \otimes x)) \\ &= \delta_{U,\epsilon} \tau(x)(\mathrm{id} \otimes \mathrm{tr})(V_{\epsilon,12}) \\ &= \delta_{U,\epsilon} \tau(x) 1_A \,.\end{aligned}$$

So ψ is trace preserving and hence it extends to a $*$-homomorphism $\psi : M \to \mathrm{M}_n(\mathbb{C}) \otimes M$.

Step 2. Define a functor $F : \mathrm{UCorep}_{\mathrm{fin}}(A^{\mathrm{coop}}) \to \mathrm{Bimod}(Q \subset Q_1)$ such that if V is a finite dimensional corepresentation of A^{coop} and ψ the map associated with it in Step 1, we have $F(V) = \mathcal{H}(\psi)$. If $T \in \mathrm{Hom}(V_1, V_2)$ is an intertwiner, we set $F(T) = T \otimes \mathrm{id} : H_{V_1} \otimes \mathrm{L}^2(M) \to H_{V_2} \otimes \mathrm{L}^2(M)$. Then F is fully faithful tensor C*-functor.

Proof of Step 2. It is obvious that F is faithful. In order to show that F is full, let $V_1 \in A \otimes \mathrm{M}_m(\mathbb{C})$,$V_2 \in A \otimes \mathrm{M}_n(\mathbb{C})$ be finite dimensional corepresentations of A^{coop}. Denote by ψ_1, ψ_2 the maps associated with V_1 and V_2, respectively. Let $T : \mathbb{C}^m \otimes \mathrm{L}^2(M) \to \mathbb{C}^n \otimes \mathrm{L}^2(M)$ be an intertwiner from $\mathcal{H}(F(V_1))$ to $\mathcal{H}(F(V_2))$. Then $T \in \mathrm{B}(\mathbb{C}^m, \mathbb{C}^n) \otimes M$ satisfies

$$T(1 \otimes x) = T\psi_1(x) = \psi_2(x) T = (1 \otimes x) T \quad \text{for all } x \in N \,.$$

Hence, $T \in \mathrm{B}(\mathbb{C}^m, \mathbb{C}^n) \otimes 1$. So, for any irreducible corepresentation U of $\widehat{A}$, we have

$$\begin{aligned} V_{2,12} T_{23} &= U_{13}^* U_{13} V_{2,12} T_{23} \\ &= U_{13}^* \psi_2(U) T_{23} \\ &= U_{13}^* T_{23} \psi_1(U) \\ &= T_{23} U_{13}^* U_{13} V_{1,12} \\ &= T_{23} V_{1,12} \,.\end{aligned}$$

So T comes from an intertwiner from V_1 to V_2. This shows that F is full.

For an intertwiner $T \in \mathrm{Hom}(V_1, V_2)$ we have $F(T^*) = F(T)^*$, so F is a C*-functor.

If V_1,V_2 are finite dimensional corepresentations of A^{coop}, ψ_1, ψ_2 and ψ denote the maps associated with V_1, V_2 and $V_{1,12}V_{2,13} = V_1 \boxtimes V_2$ respectively, then $(\text{id} \otimes \psi)(U) = U_{14}V_{1,12}V_{2,13} = (\text{id} \otimes \psi_2) \circ \psi_1(U)$, for every irreducible corepresentation $U \in \mathrm{B}(H_U) \otimes \widehat{A}$ of $\widehat{A}$. So $\psi = (\text{id} \otimes \psi_2) \circ \psi_1$. We obtain $F(V_1) \otimes_M F(V_2) \cong F(V_1 \boxtimes V_2)$ and this unitary isomorphism is natural in V_1 and V_2. Hence F is a tensor C*-functor.

Step 3. F is essentially surjective.

Proof of Step 3. Let $\mathcal{H}$ be a finite index bimodule in $\text{Bimod}(M \subset \widehat{A} \ltimes M)$. Write $\mathcal{H} \cong \mathcal{H}(\psi)$ for some $\psi : M \to p(\mathrm{M}_n(\mathbb{C}) \otimes M)p$ satisfying $p \in (1 \otimes N)' \cap (\mathrm{M}_n(\mathbb{C}) \otimes M)$ and $\psi(x) = p(1 \otimes x)$ for all $x \in N$. Since $N \subset M$ is irreducible, we have $p \in \mathrm{M}_n(\mathbb{C}) \otimes 1$, so we may assume that $p = 1$. For an irreducible corepresentation U of $\widehat{A}$, by the same calculation as in Step 1, we obtain

$$(\text{id} \otimes \psi)(U)U_{13}^*\alpha_U(x) = \alpha_U(x)(\text{id} \otimes \psi)(U)U_{13}^*\,,$$

for all $x \in N$. Since N is linearly generated by the coefficients of $\alpha_U(N)$, it follows that

$$(\text{id} \otimes \psi)(U)U_{13}^* = V_{U,12}$$

for some element in $V_U \in \mathrm{B}(H_U) \otimes \mathrm{M}_n(\mathbb{C}) \subset A \otimes \mathrm{M}_n(\mathbb{C})$. Let

$$V = \boxplus_{U \text{ irr. corep. of } \widehat{A}} V_U \in A \otimes \mathrm{M}_n(\mathbb{C})\,.$$

We show that V is a corepresentation of A^{coop}, i.e. that $(\Delta \otimes \text{id})(V) = V_{13}V_{23}$. It suffices to show for any irreducible corepresentations U_1, U_2, U_3 and any intertwiner $T \in \text{Hom}(U_1, U_2 \boxtimes U_3)$ that we have

$$V_{U_2,13}V_{U_3,23}(T \otimes \text{id}) = (T \otimes \text{id})(V_{U_1})\,.$$

Indeed, we have

$$\begin{aligned}(T \otimes \text{id})(V_{U_1} \otimes 1) &= (T \otimes \text{id})(\text{id} \otimes \psi)(U_1)U_{1,13}^* \\ &= (\text{id} \otimes \psi)(U_2)_{134}(\text{id} \otimes \psi)(U_3)_{234}(U_{2,14}U_{3,24})^*(T \otimes \text{id}) \\ &= (\text{id} \otimes \psi)(U_2)_{134}V_{3,23}U_{2,14}^*(T \otimes \text{id}) \\ &= (\text{id} \otimes \psi)(U_2)_{134}U_{2,14}^*V_{3,23}(T \otimes \text{id}) \\ &= V_{2,13}V_{3,23}(T \otimes \text{id})\,.\end{aligned}$$

This shows that V is a corepresentation of A and $\mathcal{H}(\psi) = F(V)$. □

Chapter 5

On the classification of free Bogoliubov crossed product von Neumann algebras by the integers

This chapter is based on [183]. We consider crossed product von Neumann algebras arising from free Bogoliubov actions of $\mathbb{Z}$. We describe several presentations of them as amalgamated free products and cocycle crossed products and give a criterion for factoriality. A number of isomorphism results for free Bogoliubov crossed products are proved, focusing on those arising from almost periodic representations. We complement our isomorphism results by rigidity results yielding non-isomorphic free Bogoliubov crossed products and by a partial characterisation of strong solidity of a free Bogoliubov crossed products in terms of properties of the orthogonal representation from which it is constructed

5.1 Introduction

With an orthogonal representation (H, π) of a discrete group G, Voiculescu's free Gaussian functor associates an action of G on the free group factor $\Gamma(H)'' \cong L\mathbb{F}_{\dim H}$ (see Section 5.2.1 and [226, Section 2.6]). An action arising this way is called a *free Bogoliubov action* of G. The associated free Bogoliubov crossed product von Neumann algebras $\Gamma(H)'' \rtimes G$, also denoted by $\Gamma(H, G, \pi)''$, were studied by several authors [194, 112, 105, 106]. Note that in [194, Section 7] free Bogoliubov crossed products with $\mathbb{Z}$ appear under the name of *free Krieger algebras* (see also [193, Section 3] and [112, Section 6]). The classification of free Bogoliubov crossed products is especially interesting because of their close relation to *free Araki-Woods factors* [192, 194]. In the context of the complete classification of free Araki-Woods factors associated with almost periodic orthogonal representations of $\mathbb{R}$ [192, Theorem 6.6], already the classification of the corresponding class of free Bogoliubov crossed products becomes an attractive problem.

Popa initiated his *deformation/rigidity theory* in 2001 [165, 164, 166, 167, 171]. During the past decade this theory enabled him to prove a large number of non-isomorphism results for von Neumann algebras and to calculate many of their invariants. In particular, he obtained the first rigidity results for *group measure space II_1 factors* in [166, 167]. Moreover, he obtained the first calculations of *fundamental groups* not equal to $\mathbb{R}_{>0}$ in [164] and of outer automorphisms groups in [117]. Further developments in the deformation/rigidity theory led Ozawa and Popa to the discovery of *II_1 factors with a unique Cartan subalgebra* in [155, 156]. Also *W^*-superrigidity* theorems for group von Neumann algebras [118, 27] and group measure space II_1 factors [177, 173, 174, 114] were proved by means of deformation/rigidity techniques. In the context of free Bogoliubov actions Popa's techniques were applied too. In [165, Section 6], Popa introduced the *free malleable deformation* of free Bogoliubov crossed products. This lead in [108] and, using the work of Ozawa-Popa, in [112, 111, 106] to several structural results and rigidity theorems for free Araki-Woods factors and free Bogoliubov crossed products. We use the main result of [112] in order to obtain certain non-isomorphism results for free Bogoliubov crossed products.

In the cause of the deformation/rigidity theory, absence of Cartan algebras and primeness were studied too. The latter means that a given II_1 factor has no decomposition as a tensor product of two II_1 factors. Ozawa introduced in [153] the notion of *solid* II_1 factors, that is II_1 factors M such that for all diffuse von Neumann subalgebras $A \subset M$ the relative commutant $A' \cap M$ is amenable. In [170], Popa used his deformation/rigidity techniques in order to prove solidity of the free group factors, leading to the discovery of *strongly solid* II_1 factors in [155, 156]. A II_1 factor M is strongly solid if for all amenable, diffuse von Neumann subalgebras $A \subset M$, its normaliser $\mathcal{N}_M(A)''$ is amenable too. We extend the results of [112] on strong solidity of certain free Bogoliubov crossed products and point out a class of non-solid free Bogoliubov crossed products.

Opposed to non-isomorphism results obtained in Popa's deformation/rigidity theory, there are two known sources of isomorphism results for von Neumann algebras. First, the *classification of injective von Neumann algebras* by Connes [52] shows that all group measure space II_1 factors $\mathrm{L}^\infty(X) \rtimes G$ associated with free, ergodic, probability measure preserving actions $G \curvearrowright X$ are isomorphic to the hyperfinite II_1 factor R. By [151, 55], if $H \curvearrowright Y$ is another free, ergodic, probability measure preserving action of an amenable group, then these actions are *orbit equivalent*, meaning that there is a probability measure preserving isomorphism $\Delta : X \to Y$ such that $\Delta(G \cdot x) = H \cdot \Delta(x)$ for almost every $x \in X$. By a result of Singer [196], this means that there is an isomorphism $\mathrm{L}^\infty(X) \rtimes G \cong \mathrm{L}^\infty(Y) \rtimes G$ sending $\mathrm{L}^\infty(X)$ to $\mathrm{L}^\infty(Y)$.

The second source of unexpected isomorphism results for von Neumann algebras is *free probability theory* as it was initiated by Voiculescu [223]. More specifically, we use the work of Dykema on *interpolated free group factors* and *amalgamated free products*. Interpolated free group factors were independently introduced by Dykema [73] and Rădulescu [181]. If M is a II_1 factor, the *amplification of M by t* is $M^t = p(\mathrm{M}_n(\mathbb{C}) \otimes M)p$, where $p \in \mathrm{M}_n(\mathbb{C}) \otimes M$ is a projection of non-normalised trace $\mathrm{Tr} \otimes \tau(p) = t$. It does not depend on the specific choice of n and p. The interpolated free group factors can be defined by

$$\mathrm{L}\mathbb{F}_r = (\mathrm{L}\mathbb{F}_n)^t \,, \text{ where } r = 1 + \frac{n-1}{t^2} \,, \text{ for some } t > 1 \text{ and } n \in \mathbb{N}_{\geqslant 2} \,.$$

Dykema's first result on free products of von Neumann algebras in [72] says that $\mathrm{L}(\mathbb{F}_n) * R \cong \mathrm{L}(\mathbb{F}_{n+1})$ for any natural number n. He developed his techniques in [73, 71, 74, 75] arriving in [76] at a description of arbitrary amalgamated free products $A *_D B$ with respect to trace-preserving conditional expectations, where A and B are tracial direct sums of hyperfinite von Neumann algebras and interpolated free group factors and the amalgam D is finite dimensional.

We combine the work of Dykema with a result on factoriality of certain amalgamated free products. The first such results for proper amalgamated free products were obtained by Popa in [162, Theorem 4.1], followed by several results of Ueda in the non-trace preserving setting [209, 210, 211, 208]. We will use a result of Houdayer-Vaes [113, Theorem 5.8], which allows for a particularly easy application in this chapter.

Section 5.3 treats the structure of free Bogoliubov crossed products. We obtain several different representations of free Bogoliubov crossed products associated with almost periodic orthogonal representations of $\mathbb{Z}$ in Theorem 5.3.3 and Proposition 5.3.7. We calculate the normaliser and the quasi-normaliser of the canonical abelian von Neumann subalgebra of a free Bogoliubov crossed product in Corollary 5.3.9 and address the question of factoriality of free Bogoliubov crossed products in Corollary 5.3.10. Most of the results in this section are probably folklore.

In Section 5.4, we obtain isomorphism results for free Bogoliubov crossed products associated with almost periodic orthogonal representations. In particular, we classify free Bogoliubov crossed products associated with non-faithful orthogonal representations of $\mathbb{Z}$ in terms of the dimension of the representation and the index of its kernel. They are tensor products of a diffuse abelian von Neumann algebra with an interpolated free group factor.

Theorem 5.A (See Theorem 5.4.3). *Let (π, H) be a non-faithful orthogonal representation of $\mathbb{Z}$ of dimension at least 2. Let $r = 1 + (\dim \pi - 1)/[\mathbb{Z} : \ker \pi]$. Then*

$$\Gamma(H, \mathbb{Z}, \pi)'' \cong \mathrm{L}^\infty([0,1]) \overline{\otimes} \mathrm{L}\mathbb{F}_r ,$$

by an isomorphism carrying the subalgebra $\mathrm{L}\mathbb{Z}$ of $\Gamma(H, \mathbb{Z}, \pi)''$ onto $\mathrm{L}^\infty([0,1]) \otimes \mathbb{C}^{[\mathbb{Z}:\ker \pi]}$.

For general almost periodic orthogonal representations of $\mathbb{Z}$ we can prove that the isomorphism class of the free Bogoliubov crossed product depends at most on their dimension and on the concrete subgroup of S^1 generated by the eigenvalues of their complexification. More generally, we have the following result.

Theorem 5.B (See Theorem 5.4.2). *The isomorphism class of the free Bogoliubov crossed product associated with an orthogonal representation π of $\mathbb{Z}$ with almost periodic part π_{ap} depends at most on the weakly mixing part of π, the dimension of π_{ap} and the concrete embedding into S^1 of the group generated by the eigenvalues of the complexification of π_{ap}.*

In contrast to the preceding result, we show later that representations with almost periodic parts of different dimension can be non-isomorphic.

Theorem 5.C (See Theorem 5.5.1 and Theorem 5.6.4). *If λ denotes the left regular orthogonal representation of $\mathbb{Z}$ and π denotes some one dimensional orthogonal representation, then*

$$\Gamma(\ell^2(\mathbb{Z}) \oplus \mathbb{C}, \mathbb{Z}, \lambda \oplus \pi)'' \cong \Gamma(\ell^2(\mathbb{Z}), \mathbb{Z}, \lambda)'' \cong \mathrm{L}(\mathbb{F}_2) \not\cong \Gamma(\ell^2(\mathbb{Z}) \oplus \mathbb{C}^2, \mathbb{Z}, \lambda \oplus 2 \cdot \mathbb{1})'' .$$

The next results shows, however, that there are representations whose complexifications generate isomorphic, but different subgroups of S^1 and their free Bogoliubov crossed products are isomorphic nevertheless.

Theorem 5.D (See Corollary 5.4.5). *All faithful two dimensional representations of $\mathbb{Z}$ give rise to isomorphic free Bogoliubov crossed products.*

Inspired by the connection between free Bogoliubov crossed products and cores of Araki-Woods factors, and classification results for free Araki-Woods factors

[192], Shlyakhtenko asked at the 2011 conference on von Neumann algebras and ergodic theory at IHP, Paris, whether for an orthogonal representation $(\pi_{\mathbb{R}}, H_{\mathbb{R}})$ of $\mathbb{Z}$ the isomorphism class of $\Gamma(H_{\mathbb{R}}, \mathbb{Z}, \pi_{\mathbb{R}})''$ is completely determined by the representation $\bigoplus_{n \geqslant 1} \pi_{\mathbb{R}}^{\otimes n}$ up to amplification. The present chapter shows that this is not the case. We discuss other possibilities of how a classification of free Bogoliubov crossed products could look like and put forward the following conjecture in the almost periodic case.

Conjecture 5.A (See Conjecture 5.4.6)**.** *The abstract isomorphism class of the subgroup generated by the eigenvalues of the complexification of an infinite dimensional, faithful, almost periodic orthogonal representation of $\mathbb{Z}$ is a complete invariant for isomorphism of the associated free Bogoliubov crossed product.*

In Section 5.5, we describe strong solidity and solidity of a free Bogoliubov crossed product $\Gamma(H, \mathbb{Z}, \pi)''$ in terms of properties of π. The main result of [112] on strong solidity of free Bogoliubov crossed products is combined with ideas of Ioana [114] in order to obtain a bigger class of strongly solid free Bogoliubov crossed products of $\mathbb{Z}$.

Theorem 5.E (See Theorem 5.5.2)**.** *Let (π, H) be the direct sum of a mixing representation and a representation of dimension at most one. Then $\Gamma(H, \mathbb{Z}, \pi)''$ is strongly solid.*

Orthogonal representations that have an invariant subspace of dimension two give rise to free Bogoliubov crossed products, which are obviously not strongly solid. In particular, all almost periodic orthogonal representations are part of this class of representations. The next theorem describes a more general class of representations of $\mathbb{Z}$ that give rise to non-solid free Bogoliubov crossed products. If (π, H) is a representation of $\mathbb{Z}$, we say that a non-zero subspace $K \leqslant H$ is rigid if there is a sequence $(n_k)_k$ in $\mathbb{Z}$ such that $\pi(n_k)|_K$ converges to id_K strongly as $n_k \to \infty$.

Theorem 5.F (See Theorem 5.5.4)**.** *If the orthogonal representation (π, H) of $\mathbb{Z}$ has a rigid subspace of dimension two, then the free Bogoliubov crossed product $\Gamma(H, \mathbb{Z}, \pi)''$ is not solid.*

We make the conjecture that this theorem describes all non-solid free Bogoliubov crossed products of the integers.

Conjecture 5.B (See Conjecture 5.5.5)**.** *If (π, H) is an orthogonal representation of $\mathbb{Z}$, then the following are equivalent.*

- $\Gamma(H, \mathbb{Z}, \pi)''$ *is solid.*

- $\Gamma(H, \mathbb{Z}, \pi)''$ *is strongly solid.*
- π *has no rigid subspace of dimension two.*

In Section 5.6, we prove a rigidity result for free Bogoliubov crossed products associated with orthogonal representations having at least a two dimensional almost periodic part. Due to the lack of invariants for bimodules over abelian von Neumann algebras, we can obtain only some non-isomorphism results.

Theorem 5.G (See Theorem 5.6.4)**.** *No free Bogoliubov crossed product associated with a representation in the following classes is isomorphic to a free Bogoliubov crossed product associated with a representation in the other classes.*

- *The class of representations* $\lambda \oplus \pi$*, where* λ *is the left regular representation of* $\mathbb{Z}$ *and* π *is a faithful almost periodic representation of dimension at least* 2*.*
- *The class of representations* $\lambda \oplus \pi$*, where* λ *is the left regular representation of* $\mathbb{Z}$ *and* π *is a non-faithful almost periodic representation of dimension at least* 2*.*
- *The class of representations* $\rho \oplus \pi$*, where* ρ *is a representation of* $\mathbb{Z}$ *whose spectral measure* μ *and all of its convolutions* μ^{*n} *are non-atomic and singular with respect to the Lebesgue measure on* S^1 *and* π *is a faithful almost periodic representation of dimension at least* 2*.*
- *The class of representations* $\rho \oplus \pi$*, where* ρ *is a representation of* $\mathbb{Z}$ *whose spectral measure* μ *and all of its convolutions* μ^{*n} *are non-atomic and singular with respect to the Lebesgue measure and* π *is a non-faithful almost periodic representation of dimension at least* 2*.*
- *Faithful almost periodic representations of dimension at least* 2*.*
- *Non-faithful almost periodic representations of dimension at least* 2*.*
- *The class of representations* $\rho \oplus \pi$*, where* ρ *is mixing and* $\dim \pi \leqslant 1$*.*

5.2 Preliminaries

5.2.1 Orthogonal representations of $\mathbb{Z}$ and free Bogoliubov shifts

With a real Hilbert space H, Voiculescu's *free Gaussian functor* associates a von Neumann algebra $\Gamma(H)'' \cong \mathrm{L}\mathbb{F}_{\dim H}$ [226]. For every vector $\xi \in H$, we have a self-adjoint element $s(\xi) \in \Gamma(H)''$ and $\Gamma(H)''$ is generated by these elements. If $\xi, \eta \in H$ are orthogonal then $s(\xi) + is(\eta)$ is an element with circular distribution with respect to the trace on $\Gamma(H)''$. In particular, the polar decomposition of $s(\xi) + is(\eta)$ equals $a \cdot u$, where a, u are $*$-free from each other, a has a quarter-circular distribution and u is a Haar unitary. The free Gaussian construction $\Gamma(H)''$ acts by construction on the full Fock space $\mathbb{C}\Omega \oplus \bigoplus_{n \geqslant 1} H^{\otimes n}$ where Ω is called the vacuum vector. It is cyclic and separating for $\Gamma(H)''$ and $\Gamma(H)''\Omega \supset H^{\otimes_{\mathrm{alg}} n}$ for all $n \in \mathbb{N}$. Hence, for $\xi_1 \otimes \cdots \otimes \xi_n \in H^{\otimes_{\mathrm{alg}} n}$, there is a unique element $W(\xi_1 \otimes \cdots \otimes \xi_n) \in \Gamma(H)''$ such that $W(\xi_1 \otimes \cdots \otimes \xi_n)\Omega = \xi_1 \otimes \cdots \otimes \xi_n$.

The free Gaussian construction is functorial for isometries, so that an orthogonal representation (π, H) of a group G yields a trace preserving action $G \curvearrowright \Gamma(H)''$, which is completely determined by $g \cdot s(\xi) = s(\pi(g)\xi)$. If $\xi_1 \otimes \cdots \otimes \xi_n \in H^{\otimes_{\mathrm{alg}} n}$ and $g \in G$, then $g \cdot W(\xi_1 \otimes \cdots \otimes \xi_n) = W(\pi(g)\xi_1 \otimes \cdots \otimes \pi(g)\xi_n)$.

An action obtained by the free Gaussian functor is called *free Bogoliubov action.* If $G \curvearrowright \Gamma(H)''$ is the free Bogoliubov action associated with (π, H), then the representation of G on $\mathrm{L}^2(\Gamma(H)'') \ominus \mathbb{C} \cdot 1$ is isomorphic with $\bigoplus_{n \geqslant 1} \pi^{\otimes n}$. The associated von Neumann algebraic crossed product $\Gamma(H)'' \rtimes G$ of a free Bogoliubov action is denoted by $\Gamma(H, G, \pi)''$. If there is no confusion possible, we denote $\Gamma(H, G, \pi)''$ by M_π and the algebra $\mathrm{L}G \subset \Gamma(H, G, \pi)''$ by A_π.

An orthogonal representation (π, H) is called *almost periodic* if it is the direct sum of finite dimensional representations. It is called *periodic* if the map π has a kernel of finite index in G. We call π *weakly mixing*, if it has no finite dimensional subrepresentation. Every orthogonal representation (π, H) is the direct sum of an almost periodic representation $(\pi_{\mathrm{ap}}, H_{\mathrm{ap}})$ and a weakly mixing representation $(\pi_{\mathrm{wm}}, H_{\mathrm{wm}})$.

Spectral theory says that unitary representations π of $\mathbb{Z}$ correspond to pairs (μ, N), where μ is a Borel measure on S^1 and N is a function with values in $\mathbb{N} \cup \{\infty\}$ called the multiplicity function of π. The measure μ and the equivalence class of N up to changing it on μ-negligible sets are uniquely determined by π. Given any orthogonal representation (π, H) of $\mathbb{Z}$, denote by $(\pi_{\mathbb{C}}, H_{\mathbb{C}})$ its complexification. Note that a pair (μ, N) as above is associated with a complexification of an orthogonal representation if and only if μ and N are invariant under complex conjugation on $\mathrm{S}^1 \subset \mathbb{C}$. An orthogonal representation (π, H) is weakly mixing if and only if μ has no atoms. It is almost periodic if and only if the measure associated with $(\pi_{\mathbb{C}}, H_{\mathbb{C}})$ is completely atomic. In this case the atoms of μ and the function N together form the *multiset of eigenvalues with multiplicity* of $\pi_{\mathbb{C}}$. Up to isomorphism, an almost periodic representation π is uniquely determined by this multiset.

5.2.2 Rigid subspaces of group representations

A rigid subspace of an orthogonal representation (π, H) of a discrete group G is a non-zero Hilbert subspace $K \leqslant H$ such that there is a sequence $(g_n)_n$ of elements in G tending to infinity that satisfies $\pi(g_n)\xi \longrightarrow \xi$ as $n \to \infty$ for all $\xi \in K$. Note that this terminology is borrowed from ergodic theory and has nothing to do with property (T).

A representation π without any rigid subspace is called *mildly mixing*. The main source of mildly mixing representations of groups are mildly mixing actions [190]. A probability measure preserving action $G \curvearrowright (X, \mu)$ has a rigid factor if there is a Borel subset $B \subset X$, $0 < \mu(B) < 1$ such that $\liminf_{g \to \infty} \mu(B \Delta gB) = 0$. We say that $G \curvearrowright (X, \mu)$ is mildly mixing if it has no rigid factor.

Proposition 5.2.1. *Let $G \curvearrowright (X, \mu)$ be a probability measure preserving action of a group G. Then the Koopman representation $G \curvearrowright \mathrm{L}^2{}_0(X, \mu)$ is mildly mixing if and only if $G \curvearrowright (X, \mu)$ is mildly mixing.*

Proof. First assume that the Koopman representation is mildly mixing and take $B \subset X$ a Borel subset such that there is a sequence $(g_n)_n$ in G going to infinity that satisfies $\mu(B \Delta g_n B) \to 0$. Consider the function $\xi = \mu(B) \cdot 1 - 1_B \in \mathrm{L}^2{}_0(X, \mu)$. Then

$$\|\xi - g_n \xi\|_2^2 = \|1_{g_n B} - 1_B\|_2^2 = \mu(B \Delta g_n B) \to 0\,.$$

By mild mixing of $G \curvearrowright \mathrm{L}^2{}_0(X, \mu)$, it follows that $\xi = 0$, so $\mu(B) \in \{0, 1\}$. Hence $G \curvearrowright (X, \mu)$ is mildly mixing.

For the converse implication assume that there is a sequence $(g_n)_n$ in G tending to infinity such that there is a unit vector $\xi \in \mathrm{L}^2{}_0(X, \mu)$ that satisfies $g_n \xi \to \xi$. We have to show that $G \curvearrowright (X, \mu)$ has a rigid factor. Replacing ξ by its real part, we may assume that it takes only real values. For $\delta > 0$ define $A_\delta = \{x \mid \xi(x) \geqslant \delta\}$ and $B_\delta = \{x \mid \xi(x) > \delta\}$. Since $\int_X \xi(x) \mathrm{d}\mu(x) = 0$, there is some $\delta > 0$ such that $0 < \mu(A_\delta) < 1$.

Take $\varepsilon > 0$. We have $\bigcap_{\delta' < \delta} B_{\delta'} = A_\delta$, so that we can choose $\delta' < \delta$ such that $\mu(B_{\delta'} \backslash A_\delta) < \varepsilon/4$. Take $N \in \mathbb{N}$ such that for all $n \geqslant N$ we have $\|\xi - g_n \xi\| < (\delta - \delta') \cdot \varepsilon/4$. Then for all $n \geqslant N$, we have

$$\begin{aligned}
\mu(A_\delta \Delta g_n A_\delta) &= \mu(A_\delta \backslash g_n A_\delta) + \mu(A_\delta \backslash g_n^{-1} A_\delta) \\
&< \mu(A_\delta \backslash g_n B_{\delta'}) + \mu(A_\delta \backslash g_n^{-1} B_{\delta'}) + \frac{\varepsilon}{2} \\
&\leqslant \frac{1}{(\delta - \delta')^2} \left(\int_{A_\delta \backslash g_n B_{\delta'}} |\xi(x) - g_n \xi(x)|^2 \mathrm{d}x + \right. \\
&\qquad \left. \int_{A_\delta \backslash g_n^{-1} B_{\delta'}} |\xi(x) - g_n^{-1} \xi(x)|^2 \mathrm{d}x \right) + \frac{\varepsilon}{2} \\
&\leqslant \frac{2}{(\delta - \delta')^2} \int_X |\xi(x) - g_n \xi(x)|^2 \mathrm{d}\mu(x) + \frac{\varepsilon}{2} \\
&< \varepsilon \, .
\end{aligned}$$

It follows that $\mu(A_\delta \Delta g_n A_\delta) \to 0$ as $n \to \infty$. So $G \curvearrowright (X, \mu)$ is not mildly mixing. □

5.2.3 Bimodules over von Neumann algebras

Let M, N be von Neumann algebras. An M-N-bimodule is a Hilbert space $\mathcal{H}$ with a normal $*$-representation of $\lambda : M \to \mathscr{B}(\mathcal{H})$ and a normal anti-$*$-representation $\rho : N \to \mathscr{B}(H)$ such that $\lambda(x)\rho(y) = \rho(y)\lambda(x)$ for all $x \in M$, $y \in N$. If M, N are tracial, then we have ${}_M\mathcal{H} \cong {}_M(\mathrm{L}^2(M) \otimes \ell^2(\mathbb{N})^*)p$ with $p \in M \otimes \mathscr{B}(\ell^2(\mathbb{N}))$. The left dimension $\dim_{M-} \mathcal{H}$ of ${}_M\mathcal{H}$ is $(\tau_M \otimes \mathrm{Tr})(p)$ by definition. Similarly, we define the right dimension $\dim_{-N} \mathcal{H}$ of $\mathcal{H}_N$. We say that ${}_M\mathcal{H}_N$ is left finite, if it has finite left dimension, we call it right finite if it has finite right dimension and we say that $\mathcal{H}$ is a finite index M-N-bimodule, if its left and right dimension are both finite.

If $A, B \subset M$ are abelian von Neumann algebras and ${}_A\mathcal{H}_B \subset \mathrm{L}^2(M)$ is a finite index bimodule, then there are non-zero projections $p \in A, q \in B$, a finite index inclusion $\phi : pA \to qB$ and a non-zero partial isometry $v \in pMq$ such that $av = v\phi(a)$ for all $a \in pA$. Since ϕ is a finite index inclusion, we can cut down p and q so as to assume that ϕ is an isomorphism.

5.2.4 The measure associated with a bimodule over an abelian von Neumann algebra

We describe bimodules over abelian von Neumann algebras, as in [53, V. Appendix B]. Compare also with [142, Section 3] concerning our formulation. Let $A \cong \mathrm{L}^\infty(X, \mu)$ be an abelian von Neumann algebra and ${}_A\mathcal{H}_A$ an A-A-bimodule such that $\lambda, \rho : A \to \mathscr{B}(\mathcal{H})$ are faithful. Then the two inclusions $\lambda, \rho : A \to \mathscr{B}(\mathcal{H})$ generate an abelian von Neumann algebra $\mathcal{A}$. Writing $[\nu]$ for the class of a measure ν and p_1, p_2 for the projections on the two factors of $X \times X$, we can identify $\mathcal{A} \cong \mathrm{L}^\infty(X \times X, \nu)$ where $[\nu]$ is subject to the condition $(p_1)_*([\nu]) = (p_2)_*([\nu]) = [\mu]$. We can disintegrate $\mathcal{H}$ with respect to ν and obtain a decomposition $\mathcal{H} = \int_{X \times X}^{\oplus} \mathcal{H}_{x_1,x_2} \mathrm{d}\nu(x_1, x_2)$. Let $N : X \times X \to \mathbb{N} \cup \{\infty\}$ be the dimension function $\mathcal{H}_{x_1,x_2} \mapsto \dim_{\mathbb{C}} \mathcal{H}_{x_1,x_2}$. Then N is unique up to changing it on ν-negligible sets and the triple $(X, [\nu], N)$ is a conjugacy invariant for ${}_A\mathcal{H}_A$ in the following sense. Let $(X, [\nu_X], N_X)$ and $(Y, [\nu_Y], N_Y)$ be triples as before associated with bimodules $\mathcal{H}_X$ and $\mathcal{H}_Y$ over $A = \mathrm{L}^\infty(X, \mu_x)$ and $B = \mathrm{L}^\infty(Y, \mu_y)$, respectively. A measurable isomorphisms $\Delta : (X, [\mu_X]) \to (Y, [\mu_Y])$ such that $(\Delta \times \Delta)_*([\nu_X]) = [\nu_Y]$ and $N_Y \circ (\Delta \times \Delta) = N_X$ ν_Y-almost everywhere induces an isomorphism $\theta : A \to B$ and a unitary isomorphism $U : \mathcal{H}_X \to \mathcal{H}_Y$ satisfying

$$U\lambda_X(a) = \lambda_Y(\theta(a))U \text{ and } U\rho_X(a) = \rho_Y(\theta(a))U \text{ for all } a \in A\,.$$

Moreover, any such pair (U, θ) arises this way. The proof of this fact works similar to that of [142].

Let ${}_A\mathcal{H}_A$ be an A-A-bimodule and identify $A \cong \mathrm{L}^\infty(X, \mu)$. Denote by $(X, [\nu], N)$ the spectral invariant of ${}_A\mathcal{H}_A$ as described in the previous paragraph. If $p = 1_Y \in A$ is a non-zero projection, then it follows right away that the spectral invariant associated with ${}_{pA}(p\mathcal{H}p)_{pA}$ equals $(Y, [\nu|_{Y \times Y}], N|_{Y \times Y})$.

Let $\mathbb{Z} \curvearrowright P$ be an action of $\mathbb{Z}$ on a tracial von Neumann algebra P and $M = P \rtimes \mathbb{Z}$. Let (μ, N_π) denote the spectral invariant of the representation π on $\mathrm{L}^2(P) \ominus \mathbb{C}1$ associated with the action of $\mathbb{Z}$ on P. Write $A = \mathrm{L}\mathbb{Z} \cong \mathrm{L}^\infty(\mathrm{S}^1)$, where the identification is given by the Fourier transform. We describe the spectral invariant $(\mathrm{S}^1, [\nu], N)$ of the A-A-bimodule $\mathrm{L}^2(M) \ominus \mathrm{L}^2(A)$ in terms of (π, N_π).

We first calculate the measure $\nu_{\xi \otimes \delta_n}$ on $\mathrm{S}^1 \times \mathrm{S}^1$ defined by

$$\int_{\mathrm{S}^1 \times \mathrm{S}^1} s^a t^b \,\mathrm{d}\nu_{\xi \otimes \delta_n}(s, t) = \langle u_a(\xi \otimes \delta_n)u_b, \xi \otimes \delta_n \rangle\,,$$

with $a, b \in \mathbb{Z}$, $\xi \in \mathrm{L}^2(P) \ominus \mathbb{C}1$ and $\delta_n \in \ell^2(\mathbb{Z})$ the canonical basis element associated with $n \in \mathbb{Z}$. Denote by μ_ξ the measure on S^1 defined by

$$\int_{\mathrm{S}^1} s^a \, \mathrm{d}\mu_\xi(s) = \langle \pi(a)\xi, \xi \rangle \, .$$

We obtain for $a, b \in \mathbb{Z}$, $\xi \in \mathrm{L}^2(P) \ominus \mathbb{C}1$ and $n \in \mathbb{Z}$

$$\begin{aligned}
\int_{\mathrm{S}^1 \times \mathrm{S}^1} s^a t^b \, \mathrm{d}\nu_{\xi \otimes \delta_n}(s,t) &= \langle u_a(\xi \otimes \delta_n) u_b, \xi \otimes \delta_n \rangle \\
&= \delta_{a,-b} \langle \pi(a)\xi, \xi \rangle \\
&= \delta_{a,-b} \int_{\mathrm{S}^1} s^a \, \mathrm{d}\mu_\xi(s) \\
&= \int_{\mathrm{S}^1 \times \mathrm{S}^1} s^a t^{a+b} \, \mathrm{d}(\mu_\xi \otimes \lambda)(s,t) \\
&= \int_{\mathrm{S}^1 \times \mathrm{S}^1} s^a t^b \, \mathrm{d}T_*(\mu_\xi \otimes \lambda)(s,t) \, ,
\end{aligned}$$

where $T : \mathrm{S}^1 \times \mathrm{S}^1 \to \mathrm{S}^1 \times \mathrm{S}^1 : (s,t) \mapsto (s, st)$. So $\nu_{\xi \otimes \delta_n} = T_*(\mu_\xi \otimes \lambda)$ for all $\xi \in \mathrm{L}^2(M) \ominus \mathbb{C}1$ and for all $n \in \mathbb{Z}$. It follows that $[\nu] = T_*([\mu \otimes \lambda])$.

We calculate the multiplicity function N of $\mathrm{L}^2(M) \ominus \mathrm{L}^2(A)$ in terms of N_π. Let $Y_n, n \in \mathbb{N} \cup \{\infty\}$ be pairwise disjoint Borel subsets of S^1 such that $N_\pi|_{Y_n} = n$ for all n. There is a basis $(\xi_{n,k})_{0 \leqslant k < n \in \mathbb{N} \cup \{\infty\}}$ of $\mathrm{L}^2(P) \ominus \mathbb{C}$ such that $\mu_{\xi_{n,k}}$ is has support equal to Y_n. So $\xi_{n,k} \otimes \delta_l$ with $l \in \mathbb{Z}$ and $0 \leqslant k < n \in \mathbb{N} \cup \{\infty\}$ is a basis of $\mathrm{L}^2(M) \ominus \mathrm{L}^2(A)$. Write $Z_n = T(Y_n \times \mathrm{S}^1)$. Then

$$\int_{Z_n} s^a t^b \, \mathrm{d}\nu_{\xi_{n,k} \otimes \delta_l}(s,t) = \int_{Y_n \times \mathrm{S}^1} s^a t^{a+b} \, \mathrm{d}(\mu_{\xi_{n,k}} \otimes \lambda)(s,t) \, ,$$

so the support of $\nu_{\xi_{n,k} \otimes \delta_l}$ is equal to Z_n. As a consequence, $N|_{Z_n} = n$ for all $n \in \mathbb{N} \cup \{\infty\}$. We obtain the following proposition.

Proposition 5.2.2. *Let (μ, N) be a symmetric measure with multiplicity function on S^1 having at least one atom and let π be the orthogonal representation of $\mathbb{Z}$ on $H = \mathrm{L}^2{}_{\mathbb{R}}(\mathrm{S}^1, \mu, N)$ given by $\pi(1)f = \mathrm{id}_{\mathrm{S}^1} \cdot f$. Identifying $\mathrm{L}\mathbb{Z} \cong \mathrm{L}^\infty(\mathrm{S}^1)$ via the Fourier transform, the multiplicity function of the bimodule ${}_{\mathrm{L}^\infty(\mathrm{S}^1)}\Gamma(H, \mathbb{Z}, \pi)''{}_{\mathrm{L}^\infty(\mathrm{S}^1)}$ is equal to ∞ almost everywhere.*

Proof. We have $\Gamma(H, \mathbb{Z}, \pi)'' = \Gamma(H)'' \rtimes \mathbb{Z}$, where the crossed product is taken with respect to the free Bogoliubov action of $\mathbb{Z}$ on $\Gamma(H)''$, which has $\oplus_{n \geqslant 1} \pi^{\otimes n}$

as its associated representation on $\mathrm{L}^2(\Gamma(H)'') \ominus \mathbb{C} \cdot 1$. If a is an atom of μ, then also $\overline{a}$ is one. Denote by χ_a the character of $\mathbb{Z}$ defined by $\widehat{\mathbb{Z}} \cong \mathrm{S}^1$. We have $\pi = \pi \otimes (\chi_a)^n \otimes (\chi_{\overline{a}})^n \leqslant \pi^{\otimes 2n}$. As a consequence, the multiplicity function of $\bigoplus_{n \geqslant 1} \pi^{\otimes n}$ is equal to ∞ almost everywhere. So, by the calculations preceding the remark, this is also the case for the multiplicity function of the bimodule ${}_{\mathrm{L}^\infty(\mathrm{S}^1)}\mathrm{L}^2(\Gamma(H, \mathbb{Z}, \pi)'')_{\mathrm{L}^\infty(\mathrm{S}^1)}$. □

Proposition 5.2.3. *The disintegration of* $[\nu]$ *with respect to the projection onto the first component of* $\mathrm{S}^1 \times \mathrm{S}^1$ *is given by* $[\nu] = \int [\mu * \delta_s] \, \mathrm{d}\lambda(s)$.

Proof. Let $Y, Z \subset \mathrm{S}^1$ be Borel subsets and denote by $(\mu_s)_{s \in \mathrm{S}^1}$ the constant field of measures with value μ.

$$\begin{aligned}(T_* \left(\int_{\mathrm{S}^1} \mu_s \, \mathrm{d}\lambda(s) \right))(Y \times Z) &= \int_Y \mu(Z \cdot s^{-1}) \, \mathrm{d}\lambda(s) \\ &= \int_Y \mu * \delta_s(Z) \, \mathrm{d}\lambda(s) \\ &= (\int_{\mathrm{S}^1} \mu * \delta_s \, \mathrm{d}\lambda(s))(Y \times Z) \, .\end{aligned}$$

This finishes the proof. □

5.2.5 Amalgamated free products over finite dimensional algebras

Let $\mathcal{R}_2$ denote the class of finite direct sums of hyperfinite von Neumann algebras and interpolated free group factors, equipped with a normal, faithful tracial state. In [76, Theorem 4.5], amalgamated free products of elements of $\mathcal{R}_2$ over finite dimensional tracial von Neumann subalgebras were shown to be in $\mathcal{R}_2$ again. Moreover, their *free dimension* in the sense of Dykema [75] was calculated in terms of the free dimension of the factors and of the amalgam of the amalgamated free product. We explain the free dimension and Theorem 4.5 of [76].

The free dimension of a set of generators of a von Neumann algebra $M \in \mathcal{R}_2$ is used to keep track of the parameter of interpolated free group factors. If an interpolated free group factor has a generating sets of free dimension r, then it is isomorphic to $\mathrm{L}\mathbb{F}_r$. Following [76], we define the class $\mathcal{F}_d \subset \mathcal{R}_2$, $d \in \mathbb{R}_{>0}$ as the class of von Neumann algebras

$$M = D \oplus \bigoplus_{i \in I} p_i \mathrm{L}\mathbb{F}_{r_i} \oplus \bigoplus_{j \in J} q_j \mathrm{M}_{n_j}(\mathbb{C}) \, ,$$

where

- p_i is the unit of $\mathrm{L}\mathbb{F}_{r_i}$ and q_j the unit of $\mathrm{M}_{n_j}(\mathbb{C})$,
- $t_i = \tau_M(p_i)$, $s_j = \frac{\tau_M(q_j)}{n_j}$ and D is a diffuse hyperfinite von Neumann algebra and
- $1 + \sum_i t_i^2(r_i - 1) - \sum_j s_j^2 = d$.

Theorem 4.5 of [76] says that if $M = M_1 *_A M_2$ with $M_1, M_2 \in \mathcal{R}_2$ and A a finite dimensional tracial von Neumann algebra, then $M \in \mathcal{R}_2$. Moreover, if $M_1 \in \mathcal{F}_{d_1}$, $M_2 \in \mathcal{F}_{d_2}$ and $A \in \mathcal{F}_d$, then $M \in \mathcal{F}_{d_1+d_2-d}$. We will use the following special case.

Theorem 5.2.4 (See Theorem 4.5 of [76])**.** *Let $M_1 \in \mathcal{F}_{d_1}$ and $M_2 \in \mathcal{F}_{d_2}$ and $A \in \mathcal{F}_d$ a common finite dimensional subalgebra of M_1 and M_2. If $M = M_1 *_A M_2$ is a non-amenable factor, then $M \cong \mathrm{L}\mathbb{F}_r$ with $r = d_1 + d_2 - d$.*

We will use this result in combination with a special case Theorem 5.8 of [113].

Theorem 5.2.5 (See Theorem 5.8 of [113])**.** *Let M_1, M_2 be diffuse von Neumann algebras and A a common finite dimensional subalgebra. If $\mathcal{Z}(M_1) \cap \mathcal{Z}(M_2) \cap \mathcal{Z}(A) = \mathbb{C}1$, then $M_1 *_A M_2$ is a non-amenable factor.*

5.2.6 Deformation/Rigidity

Let $A \subset M$ be an inclusion of von Neumann algebras. The normaliser of A in M, denoted by $\mathcal{N}_M(A)''$, is the von Neumann algebra generated by all unitaries $u \in M$ satisfying $uAu^* = A$. The quasi-normaliser of A in M is the von Neumann algebra $\mathrm{QN}_M(A)''$ generated by all elements $x \in M$ such that there are $a_1, \ldots, a_n$ and $b_1, \ldots, b_m$ satisfying $Nx \subset \sum_i a_i N$ and $xN \subset \sum_i N b_i$.

The following notion was introduced in [166, Theorem 2.1 and Corollary 2.3]. If M is a tracial von Neumann algebra, $A, B \subset M$ are von Neumann subalgebras, we say that A *embeds into* B *inside* M if there is a right finite A-B-subbimodule of $\mathrm{L}^2(M)$. In this case, we write $A \prec_M B$. If every A-M-subbimodule of $\mathrm{L}^2(M)$ contains a right finite A-B-subbimodule, then we say that A *fully embeds into* B *inside* M and write $A \prec_M^{\mathrm{f}} B$.

If $A, B \subset (M, \tau)$ is an inclusion of tracial von Neumann algebras, we say that A is *amenable relative to* B *inside* M, if there is an A central state φ on the basic construction $\langle M, e_B \rangle$ such that $\varphi|_M = \tau$. If A is amenable relative to an amenable subalgebra, then it is amenable itself.

We will use the following theorem from [112]. It is proven there for unital von Neumann subalgebras only, but the same proof shows, that it's true for non-unital von Neumann subalgebras.

Theorem 5.2.6 (Theorem 3.5 of [112]). *Let G be an amenable group with an orthogonal representation (π, H) and write $M = \Gamma(H, G, \pi)''$. Let $p \in M$ be a non-zero projection and $P \subset pMp$ a von Neumann subalgebra such that $P \nprec_M \mathrm{L}G$. Then $\mathcal{N}_{pMp}(P)''$ is amenable.*

Since we need full embedding of subalgebras in this chapter, let us deduce a corollary of the previous theorem.

Corollary 5.2.7 (See Theorem 3.5 of [112]). *Let G be an amenable group with an orthogonal representation (π, H) and write $M = \Gamma(H, G, \pi)''$. Let $P \subset M$ be a von Neumann subalgebra such that $\mathcal{N}_M(P)''$ has no amenable direct summand. Then $P \prec_M^{\mathrm{f}} \mathrm{L}G$.*

Proof. Take $P \subset M$ as in the statement and let us assume for a contradiction that $P \nprec_M^{\mathrm{f}} \mathrm{L}G$. Let $p \in P' \cap M$ be the maximal projection such that $pP \nprec_M \mathrm{L}G$. Then $p \in \mathcal{Z}(\mathcal{N}_M(P)'')$. By [166, Lemma 3.5], we have $\mathcal{N}_{pMp}(pP)'' \supset p\mathcal{N}_M(P)''p$. By Theorem 5.2.6, $\mathcal{N}_{pMp}(pP)''$ is amenable. So $\mathcal{N}_M(P)''$ has an amenable direct summand. This is contradiction. □

The next theorem, due to Vaes, allows us to obtain from intertwining bimodules a much better behaved finite index bimodule.

Proposition 5.2.8 (Proposition 3.5 of [216]). *Let M be a tracial von Neumann algebra and suppose that $A, B \subset M$ are von Neumann subalgebras that satisfy the following conditions.*

- *$A \prec_M B$ and $B \prec_M^{\mathrm{f}} A$.*
- *If $\mathcal{H} \leqslant \mathrm{L}^2(M)$ is an A-A bimodule with finite right dimension, then $\mathcal{H} \leqslant \mathrm{L}^2(\mathrm{QN}_M(A)'')$.*

Then there is a finite index A-B-subbimodule of $\mathrm{L}^2(M)$.

Deformation/Rigidity for amalgamated free products

We will make use of the following results, which control relative commutants in amalgamated free products.

Theorem 5.2.9 (See Theorem 1.1 of [117]). *Let $M = M_1 *_A M_2$ be an amalgamated free product of tracial von Neumann algebras and $p \in M_1$ a non-zero projection. If $Q \subset pM_1p$ is a von Neumann subalgebra such that $Q \not\prec_{M_1} A$, then $Q' \cap pMp = Q' \cap pM_1p$.*

Theorem 5.2.10 (See Theorem 6.3 in [114]). *Let $M = M_1 *_A M_2$ be an amalgamated free product of tracial von Neumann algebras and $p \in M$. Let $Q \subset pMp$ an arbitrary von Neumann subalgebra and ω a non-principal ultrafilter. Denote by B the von Neumann algebra generated by A^ω and M. One of the following statements is true.*

- *$Q' \cap (pMp)^\omega \subset B$ and $Q' \cap (pMp)^\omega \prec_{M^\omega} A^\omega$,*
- *$\mathcal{N}_{pMp}(Q)'' \prec M_i$, for some $i \in \{1, 2\}$ or*
- *Qe is amenable relative to A for some non-zero projection $e \in \mathcal{Z}(Q' \cap pMp)$.*

Also, we will need one result on relative commutants in ultrapowers.

Lemma 5.2.11 (See Lemma 2.7 in [114]). *Let M be a tracial von Neumann algebra, $p \in M$ a non-zero projection, $P \subset pMp$ and ω a non-principal ultrafilter. There is a decomposition $p = e + f$, where $e, f \in \mathcal{Z}(P' \cap (pMp)^\omega) \cap \mathcal{Z}(P' \cap pMp)$ are projections such that*

- *$e(P' \cap (pMp)^\omega) = e(P' \cap pMp)$ and this algebra is completely atomic and*
- *$f(P' \cap (pMp)^\omega)$ is diffuse.*

A tracial inclusion $B \subset M$ of von Neumann algebras is called *mixing* if for all sequences $(x_n)_n$ in the unit ball $(B)_1$ that go to 0 weakly and for all $y, z \in M \ominus B$, we have

$$\|\mathrm{E}_B(y x_n z)\|_2 \to 0 \text{ if } n \to \infty\,.$$

If a subalgebra is mixing, we can control the normaliser of algebras embedding into it.

Lemma 5.2.12 (See Lemma 9.4 in [114]). *Let $B \subset M$ be a mixing inclusion of tracial von Neumann algebras. Let $p \in M$ be a projection and $Q \subset pMp$. If $Q \prec_M B$, then $\mathcal{N}_M(Q)'' \prec_M B$.*

Finally, we will use two theorems on intertwining in amalgamated free products from the work of Ioana [114]. This theorem is stated in [114] for unital inclusions into amalgamated free products, but it remains valid in the more general case.

Theorem 5.2.13 (See Theorem 1.6 in [114])**.** *Let $M = M_1 *_A M_2$ be an amalgamated free product of tracial von Neumann algebras, $p \in M$ a projection and $Q \subset pMp$ an amenable von Neumann subalgebra. Denote by $P = \mathcal{N}_{pMp}(Q)''$ the normaliser of Q inside pMp and assume that $P' \cap (pMp)^\omega = \mathbb{C}p$ for some non-principal ultrafilter ω. Then, one of the following holds.*

- *$Q \prec A$,*
- *$P \prec M_i$, for some $i \in \{1, 2\}$ or*
- *P is amenable relative to A.*

Theorem 5.2.14 (See Theorem 9.5 in [114])**.** *Let $B \subset M$ be a mixing inclusion of von Neumann algebras. Take a non-principal ultrafilter ω, a projection $p \in M$ and let $P \subset pMp$ be a von Neumann subalgebra such that $P' \cap (pMp)^\omega$ is diffuse and $P' \cap (pMp)^\omega \prec_{M^\omega} B^\omega$. Then $P \prec_M B$.*

5.3 General structure of $\Gamma(H, \mathbb{Z}, \pi)''$

Recall that we write M_π for $\Gamma(H, \mathbb{Z}, \pi)''$. The decomposition of orthogonal representations into almost periodic and weakly mixing part, also gives rise to a decomposition of their free Bogoliubov crossed products.

Remark 5.3.1. Let (π, H) be an orthogonal representation of a discrete group G. Then

$$\Gamma(H)'' \cong \Gamma(H_{\mathrm{ap}})'' * \Gamma(H_{\mathrm{wm}})''$$

and so we get a decomposition

$$M_\pi = \Gamma(H)'' \rtimes G \cong (\Gamma(H_{\mathrm{ap}})'' \rtimes G) *_{\mathrm{L}G} (\Gamma(H_{\mathrm{wm}})'' \rtimes G) .$$

More generally, if $\pi = \bigoplus_i \pi_i$, then $M_\pi \cong *_{\mathrm{L}G,i} M_{\pi_i}$.

5.3.1 $\Gamma(H, \mathbb{Z}, \pi)''$ for almost periodic representations

If not mentioned explicitly, π denotes an almost periodic orthogonal representation of $\mathbb{Z}$ in this section. Recall that an irreducible almost periodic orthogonal representation of $\mathbb{Z}$ has dimension 1 if and only if its eigenvalue is 1 or -1. In all other cases, it has dimension 2 and its complexification has a pair of conjugate eigenvalues $\lambda, \overline{\lambda} \in \mathrm{S}^1 \backslash \{1, -1\}$.

Notation 5.3.2. We denote by $\mathrm{L}\mathbb{Z} \rtimes_\lambda \mathbb{Z}$, $\lambda \in \mathrm{S}^1$ the crossed product by the action of $\mathbb{Z}$ on $\mathrm{L}\mathbb{Z}$ where $1 \in \mathbb{Z}$ acts by multiplying the canonical generator of $\mathrm{L}\mathbb{Z}$ with λ. This is isomorphic to the crossed products $\mathrm{L}^\infty(\mathrm{S}^1) \rtimes_\lambda \mathbb{Z}$ and $\mathbb{Z} \ltimes_\lambda \mathrm{L}^\infty(\mathrm{S}^1)$, where $\mathbb{Z}$ acts on S^1 by rotation by λ. Moreover, $1 \otimes \mathrm{L}\mathbb{Z}$ is carried onto $1 \otimes \mathrm{L}\mathbb{Z}$ and $1 \otimes \mathrm{L}^\infty(\mathrm{S}^1)$, respectively, under this isomorphism.

Theorem 5.3.3. *Let π be an almost periodic orthogonal representation of $\mathbb{Z}$. Let $\lambda_i, \overline{\lambda_i}$, $0 \leqslant i < n_1 \in \mathbb{N} \cup \{\infty\}$ be an enumeration of all eigenvalues in $\mathrm{S}^1 \backslash \{1,-1\}$ of the complexification of π. Denote by n_2 and m_0 the multiplicity of -1 and 1, respectively, as an eigenvalues of π. Note that $\dim \pi = 2n_1 + n_2 + m_0$ and write $n = n_1 + n_2$, $m = n_1 + m_0$. Then*

$$\begin{aligned} M_\pi &\cong (\mathrm{L}\mathbb{F}_m \overline{\otimes} \mathrm{L}\mathbb{Z}) *_{1\otimes \mathrm{L}\mathbb{Z}} (\mathrm{L}\mathbb{F}_n \rtimes_\alpha \mathbb{Z}) \\ &\cong (\mathrm{L}\mathbb{F}_m \overline{\otimes} \mathrm{L}^\infty(\mathrm{S}^1)) *_{1\otimes \mathrm{L}^\infty(\mathrm{S}^1)} (\mathbb{F}_n \ltimes_\beta \mathrm{L}^\infty(\mathrm{S}^1)), \end{aligned}$$

where, denoting by g_i, $0 \leqslant i < n_1$, and h_i, $0 \leqslant i < n_2$, the canonical basis of $\mathbb{F}_{n_1+n_2} \cong \mathbb{F}_n$

- *$\alpha(1)$ acts on u_{g_i} by multiplication with λ_i for $0 \leqslant i < n_1$,*
- *$\alpha(1)$ acts on u_{h_i} by multiplication with -1 for $0 \leqslant i < n_2$,*
- *$\beta(g_i)$ acts on S^1 by multiplication with λ_i for $0 \leqslant i < n_1$,*
- *$\beta(h_i)$ acts on S^1 by multiplication with -1 for $0 \leqslant i < n_2$.*

Moreover, the subalgebras $\Gamma(H_\pi)'' \subset M_\pi$ and

$$\mathrm{L}(\mathbb{F}_{m+n}) \subset (\mathrm{L}\mathbb{F}_m \overline{\otimes} \mathrm{L}\mathbb{Z}) *_{1\otimes \mathrm{L}\mathbb{Z}} (\mathrm{L}\mathbb{F}_n \rtimes_\alpha \mathbb{Z})$$

are identified under this isomorphism and so are the subalgebras $\mathrm{L}\mathbb{Z}$ and $\mathrm{L}^\infty(\mathrm{S}^1)$, respectively.

Proof. If π is the trivial representation, then $M_\pi \cong \mathrm{L}\mathbb{F}_{\dim \pi} \overline{\otimes} \mathrm{L}\mathbb{Z}$. If π is the one dimensional representation with eigenvalue -1, then

$$(A_\pi \subset M_\pi) \cong (1 \otimes \mathrm{L}\mathbb{Z} \subset \mathrm{L}\mathbb{Z} \rtimes_{-1} \mathbb{Z}).$$

Let π be an irreducible two dimensional representation of $\mathbb{Z}$ with eigenvalues $\lambda, \overline{\lambda} \in \mathrm{S}^1$ of its complexification. We show that

$$M_\pi \cong (\mathrm{L}\mathbb{Z} \overline{\otimes} \mathrm{L}\mathbb{Z}) *_{1\otimes \mathrm{L}\mathbb{Z}} (\mathrm{L}\mathbb{Z} \rtimes_\lambda \mathbb{Z})$$

where the inclusion $1 \otimes \mathrm{L}\mathbb{Z} \subset (\mathrm{L}\mathbb{Z} \overline{\otimes} \mathrm{L}\mathbb{Z}) *_{1\otimes \mathrm{L}\mathbb{Z}} (\mathrm{L}\mathbb{Z} \rtimes_\lambda \mathbb{Z})$ is identified with $A_\pi \subset M_\pi$ under this isomorphism. Indeed, let $\xi, \eta \in H$ be orthogonal such that

$\xi + i\eta$ is an eigenvector with eigenvalue λ for the complexification of π. Write $c = s(\xi) + is(\eta)$. Then c is a circular element in M_π such that $\alpha_\pi(1)c = \lambda c$. Let $c = ua$ be the polar decomposition. As explained in Section 5.2.1, u is a Haar unitary and a has quarter-circular distribution and they are $*$-free from each other. Moreover, $\alpha_\pi(1)a = a$ and thus $\alpha_\pi(1)u = \lambda u$, by uniqueness of the polar decomposition. So the von Neumann algebra generated by a, u and $\mathrm{L}\mathbb{Z}$ is isomorphic to $(\mathrm{L}\mathbb{Z}\overline{\otimes}\mathrm{L}\mathbb{Z}) *_{1\otimes \mathrm{L}\mathbb{Z}} (\mathrm{L}\mathbb{Z} \rtimes \mathrm{L}\mathbb{Z})$ and A_π is identified with the subalgebra $1 \otimes \mathrm{L}\mathbb{Z}$. This gives the first isomorphism in the statement of the theorem. Since $\mathrm{L}\mathbb{Z} \rtimes_\lambda \mathbb{Z} \cong \mathbb{Z} \ltimes_\lambda \mathrm{L}^\infty(\mathrm{S}^1)$ sending $1 \otimes \mathrm{L}\mathbb{Z}$ onto $1 \otimes \mathrm{L}^\infty(\mathrm{S}^1)$ via the Fourier transform, we also obtain the second isomorphism in the statement of the theorem.

The case of a general almost periodic orthogonal representation π follows by considering its decomposition into irreducible components as in Remark 5.3.1. Indeed, denote by

$$\pi = \bigoplus_{0\leqslant i<n_1} \pi_{i,c} \oplus \bigoplus_{0\leqslant i<n_2} \pi_{i,-1} \oplus \bigoplus_{0\leqslant i<m_0} \pi_{i,1}$$

the decomposition of π into irreducible components. Here $\pi_{i,c}$ has dimension 2 with eigenvalues $\lambda_i, \overline{\lambda_i}$ of $(\pi_{i,c})_\mathbb{C}$ and $\pi_{i,-1}$ has eigenvalue -1 and $\pi_{i,1}$ is the trivial representation. Then

$$\begin{aligned}
M_\pi &\cong (\ast_{0\leqslant i<n_1} M_{\pi_{i,c}}) *_{A_\pi} (\ast_{0\leqslant i<n_2,A} M_{\pi_{i,-1}}) *_{A_\pi} (\ast_{0\leqslant i<m_0,A} M_{\pi_{i,c}}) \\
&\cong \big(\ast_{0\leqslant i<n_1,1\otimes \mathrm{L}^\infty(\mathrm{S}^1)}(\mathrm{L}\mathbb{Z} \otimes \mathrm{L}^\infty(\mathrm{S}^1)) *_{1\otimes \mathrm{L}^\infty(\mathrm{S}^1)} (\mathbb{Z} \ltimes_{\lambda_i} \mathrm{L}^\infty(\mathrm{S}^1))\big) \\
&\qquad *_{1\otimes \mathrm{L}^\infty(\mathrm{S}^1)} \big(\ast_{0\leqslant i<n_2,1\otimes \mathrm{L}^\infty(\mathrm{S}^1)}(\mathbb{Z} \ltimes_{-1} \mathrm{L}^\infty(\mathrm{S}^1))\big) \\
&\qquad *_{1\otimes \mathrm{L}^\infty(\mathrm{S}^1)} \big(\ast_{0\leqslant i<m_0,1\otimes \mathrm{L}^\infty(\mathrm{S}^1)}(\mathrm{L}\mathbb{Z} \otimes \mathrm{L}^\infty(\mathrm{S}^1))\big) \\
&\cong (\mathrm{L}\mathbb{F}_{n_1+m_0}\overline{\otimes}\mathrm{L}^\infty(\mathrm{S}^1)) *_{1\otimes \mathrm{L}^\infty(\mathrm{S}^1)} (\mathbb{F}_{n_1+n_2} \ltimes_\beta \mathrm{L}^\infty(\mathrm{S}^1)) \\
&= (\mathrm{L}\mathbb{F}_m\overline{\otimes}\mathrm{L}^\infty(\mathrm{S}^1)) *_{1\otimes \mathrm{L}^\infty(\mathrm{S}^1)} (\mathbb{F}_n \ltimes_\beta \mathrm{L}^\infty(\mathrm{S}^1))
\end{aligned}$$

and this isomorphism carries $A_\pi = \mathrm{L}\mathbb{Z}$ onto $\mathrm{L}^\infty(\mathrm{S}^1)$. □

Corollary 5.3.4. *A_π is regular inside M_π.*

Proof. By Theorem 5.3.3, we know that

$$M_\pi \cong (\mathrm{L}\mathbb{F}_m\overline{\otimes}\mathrm{L}^\infty(\mathrm{S}^1)) *_{1\otimes \mathrm{L}^\infty(\mathrm{S}^1)} (\mathbb{F}_n \ltimes_\beta \mathrm{L}^\infty(\mathrm{S}^1)),$$

and A_π is sent onto $1 \otimes \mathrm{L}^\infty(\mathrm{S}^1)$ under this isomorphism. It follows immediately that $A_\pi \subset M_\pi$ is regular. □

Note that in Theorem 5.3.3 the action of $\mathbb{F}_m$ on S^1 is not free.

Proposition 5.3.5. *Adopting the notation of Theorem 5.3.3, the relative commutant of* $L^\infty(S^1)$ *in* $(L\mathbb{F}_m \otimes L^\infty(S^1)) *_{1\otimes L^\infty(S^1)} (\mathbb{F}_n \ltimes L^\infty(S^1))$ *is* $LG\overline{\otimes}L^\infty(S^1)$, *where* $G = \mathbb{F}_m * \ker\pi$ *and* $\pi : \mathbb{F}_n \to S^1$ *sends a generator* g_i *to* λ_i *and* h_i *to* -1.

Proof. It is clear that the algebra generated by the elements u_g with $g \in G$ is part of the relative commutant of $L^\infty(S^1)$ in M_π, so we have to prove the other inclusion. Let $x \in L^\infty(S^1)' \cap M_\pi$ and write $x = \sum_{k\in Z} x_k u_k$ the Fourier decomposition with respect to the action of $\mathbb{Z}$ on $\Gamma(H_\pi)''$. Then $x_k \in L\mathbb{Z}' \cap M_\pi$, so we can assume that $x \in \Gamma(H_\pi)'' \cong L(\mathbb{F}_{m+n})$. Write $x = \sum_{g\in\mathbb{F}_{m+n}} a_g u_g$ with $a_g \in \mathbb{C}$. Since for all g the action of $\alpha(1)$ leaves $\mathbb{C}u_g$ invariant, x is fixed by α if and only if it has only coefficients in G. This proves the lemma. ☐

Corollary 5.3.6. *The von Neumann algebra* M_π *is factorial if and only if* π *is faithful.*

Proof. Let π be a non-faithful representation and take $g \in \mathbb{Z}$ such that $\pi(g) = \mathrm{id}$. Then $u_g \in L\mathbb{Z}$ is central in M_π. For the converse implication, note that π is faithful if and only if the eigenvalues of $\pi_\mathbb{C}$ generate an infinite subgroup of S^1. Any central element x of M_π must lie in $LG\overline{\otimes}L\mathbb{Z}$ and hence in $L\mathbb{Z}$, since G is a free group. Writing $L\mathbb{F}_n \rtimes \mathbb{Z} \cong \mathbb{F}_n \ltimes L^\infty(S^1)$ as in Theorem 5.3.3, the assumption implies that the action of $\mathbb{F}_n$ on $L^\infty(S^1)$ is ergodic. So $x \in \mathbb{C}1$. ☐

Using Proposition 5.3.5, we can derive a representation of M_π as a cocycle crossed product of $LG\overline{\otimes}L\mathbb{Z}$ by the group $K \subset S^1$ generated by the eigenvalues of $\pi_\mathbb{C}$. For any element $k \in K$ choose an element $g_k \in \mathbb{F}_n$ such that $\alpha(1)u_{g_k} = ku_{g_k}$. Define a G valued 2-cocycle Ω on K by

$$\Omega(k,l) = g_{kl} g_l^{-1} g_k^{-1} .$$

Then K acts on G by conjugation and on $L\mathbb{Z}$ by $k * u_1 = k \cdot u_1$. Note that if K is cyclic and infinite, then we can choose Ω to be trivial. In this case, denote by $g_1, g_2, \ldots$ a basis of $\mathbb{F}_{m+n}$ such that u_{g_1} acts by rotation on S^1 and $g_2, g_3, \ldots$ commute with A_π. We see that the elements $g_1^k g_i g_1^{-k}$, $i \geqslant 2$, $k \in \mathbb{Z}$ are a free basis of G. So K acts by shifting a free basis of G. This proves the following proposition.

Proposition 5.3.7. *There is an isomorphism* $(A_\pi \subset M_\pi) \cong (1 \otimes L^\infty(S^1) \subset K \ltimes_\Omega (LG\overline{\otimes}L^\infty(S^1)))$. *In particular, if* π *is two dimensional and faithful, then* $M_\pi \cong \mathbb{Z} \ltimes (L\mathbb{F}_\infty \overline{\otimes} L^\infty(S^1))$, *where* $\mathbb{Z}$ *acts on* $\mathbb{F}_\infty$ *by shifting the free basis and on* S^1 *by multiplication with a non-trivial eigenvalue of* $\pi_\mathbb{C}$.

5.3.2 A_π-A_π-bimodules in $\mathrm{L}^2(M_\pi)$

If π is weakly mixing, it is known [214, Proof of Theorem D.4] that every right finite A_π-A_π-bimodule is contained in $\mathrm{L}^2(A_\pi)$. More generally, we have the following proposition.

Proposition 5.3.8. *Let (π, H) be an orthogonal representation of $\mathbb{Z}$ and let $M_\pi = M_{\mathrm{ap}} *_{A_\pi} M_{\mathrm{wm}}$ be the decomposition of M_π into almost periodic and weakly mixing part. Then every right finite A_π-A_π-bimodule in $\mathrm{L}^2(M_\pi)$ lies in $\mathrm{L}^2(M_{\mathrm{ap}})$.*

Proof. By Lemma D.3 in [214], we have to prove that there is a sequence of unitaries $(u_k)_k$ in A tending to 0 weakly such that for all $x, y \in M \ominus M_{\mathrm{ap}}$ we have $\|\mathrm{E}_A(xu_n y^*)\|_2 \to 0$. It suffices to consider $x = w(\xi_1 \otimes \cdots \otimes \xi_n), y = w(\eta_1 \otimes \cdots \otimes \eta_m)$ for some $\xi_1 \otimes \cdots \otimes \xi_n H^{\otimes n}, \eta_1 \otimes \cdots \otimes \eta_m \in H^{\otimes m}$ such that at least one ξ_i and one η_j lie in H_{wm}. Take a sequence $(g_k)_k$ going to infinity in $\mathbb{Z}$ such that $\langle \pi(g_k)\xi, \eta \rangle \to 0$ for all $\xi, \eta \in H_{\mathrm{wm}}$. Then

$$\begin{aligned}
\|\mathrm{E}_A(xu_{g_k}y^*)\|_2 &= \|\mathrm{E}_A(w(\xi_1 \otimes \cdots \xi_n) w(\pi(g_k)\eta_1 \otimes \cdots \otimes \pi(g_k)\eta_m)^*) u_{g_k}\|_2 \\
&= |\tau(w(\xi_1 \otimes \cdots \xi_n) w(\pi(g_k)\eta_1 \otimes \cdots \otimes \pi(g_k)\eta_m)^*)| \\
&= \langle \xi_1 \otimes \cdots \xi_n, \pi(g_k)\eta_1 \otimes \cdots \otimes \pi(g_k)\eta_m) \rangle \\
&= \delta_{n,m} \cdot \langle \xi_1, \pi(g_k)\eta_1 \rangle \cdots \langle \xi_n, \pi(g_k)\eta_n \rangle \\
&\longrightarrow 0\,.
\end{aligned}$$

This finishes the proof. □

As an immediate consequence, we obtain the following corollaries.

Corollary 5.3.9. *Let π be an orthogonal representation of $\mathbb{Z}$. The quasi-normaliser and the normaliser of $A_\pi \subset M_\pi$ are equal to M_{ap}. In particular, $A_\pi' \cap M_\pi = \mathrm{L}G \overline{\otimes} A_\pi$, where G as defined in Proposition 5.3.5 is isomorphic to a free group.*

Proof. This follows from Proposition 5.3.8 and Corollary 5.3.4. □

Corollary 5.3.10. *If π is an orthogonal representation of $\mathbb{Z}$, then M_π is factorial if and only if π is faithful.*

Proof. This follows from Proposition 5.3.8 and Corollary 5.3.6. □

Remark 5.3.11. Note that Corollary 5.3.10 also follows directly from Theorem 5.1 of [112].

5.4 Almost periodic representations

In this section, we prove that the isomorphism class of M_π for an almost periodic orthogonal representation π of the integers depends at most on the concrete subgroup of S^1 generated by the eigenvalues of the complexification of π. We also classify non-faithful almost periodic orthogonal representations, that is periodic orthogonal representations, in terms of their kernel and their dimension.

5.4.1 Isomorphism of free Bogoliubov crossed products of almost periodic representations depends at most on the subgroup generated by the eigenvalues of their complexifications

The following lemma will be used extensively in the proof of Theorem 5.4.2.

Lemma 5.4.1. *Let S be any set and x_s, $s \in S$ a free basis of $\mathbb{F}_S$. Let $I \subset S$ and w_s, $s \in I$ be words with letters in $\{x_s \mid s \in S \backslash I\}$. Then $y_s = x_s w_s$, $s \in I$ together with $y_s = x_s$, $s \in S \backslash I$ form a basis of $\mathbb{F}_S$.*

Proof. It suffices to show that the map $\mathbb{F}_S \to \mathbb{F}_S : x_s \mapsto y_s$ has an inverse. This inverse is given by the map

$$\mathbb{F}_S \to \mathbb{F}_S : x_s \mapsto \begin{cases} x_s w_s^{-1}, & \text{if } s \in I \\ x_s, & \text{otherwise.} \end{cases}$$

□

Theorem 5.4.2. *Let π, ρ be orthogonal representations of $\mathbb{Z}$ such that*

- *their almost periodic parts have the same dimension,*
- *the eigenvalues of their complexifications generate the same concrete subgroup of S^1 and*
- *their weakly mixing parts are isomorphic.*

Then $M_\pi \cong M_\rho$ via an isomorphism that is the identity on $A_\pi = \mathrm{L}\mathbb{Z} = A_\rho$.

Proof. By the amalgamated free product decomposition $M_\pi \cong M_{\mathrm{ap}} *_{A_\pi} M_{\mathrm{wm}}$ of Remark 5.3.1, it suffices to consider almost periodic representations. Denote by G the subgroup of S^1 generated by the eigenvalues of the complexification of π. We may assume that the number of eigenvalues in $e^{2\pi i(0,\frac{1}{2})}$ of the complexification of π is larger than the one of ρ. Denote by $\lambda_i \in e^{2\pi i(0,\frac{1}{2})}$, $0 \leqslant i < n_1$, $n_1 \in \mathbb{N} \cup \{\infty\}$ and $\overline{\lambda_i}$, $0 \leqslant i < n_1$ the eigenvalues of the complexification of π that are not equal to 1 or -1. Denote by $n_2, m_0 \in \mathbb{N} \cup \{\infty\}$ the multiplicity of -1 and 1, respectively, as eigenvalues of π. By Theorem 5.3.3, we have $M_\pi \cong \mathbb{F}_{\dim \pi} \ltimes \mathrm{L}^\infty(\mathrm{S}^1)$, where $\mathbb{F}_{\dim \pi}$ has a basis consisting of

- elements x_i, $0 \leqslant i < n_1$ acting on S^1 by multiplication with λ_i,
- elements y_i, $0 \leqslant i < n_1$ acting trivially on S^1,
- elements z_i, $0 \leqslant i < n_2$ acting on S^1 by multiplication with -1 and
- elements w_i, $0 \leqslant i < m_0$ acting trivially on S^1.

Denote by $\mu_i \in e^{2\pi i(0,\frac{1}{2})}$, $0 \leqslant i < l_1 \in \mathbb{N} \cup \{\infty\}$ the non-trivial eigenvalues of the complexification of ρ that lie in the upper half of the circle and by $l_2, k_0 \in \mathbb{N} \cup \{\infty\}$ the multiplicity of -1 and 1, respectively, as an eigenvalue of ρ. Since $\dim \pi = \dim \rho$, we have $2 \cdot l_1 + l_2 + k_0 = 2 \cdot n_1 + n_2 + m_0$. We will find a new basis r_i $(0 \leqslant i < l_1)$, s_i $(0 \leqslant i < l_1 + k_0)$, t_i $(0 \leqslant i < l_2)$ of $\mathbb{F}_{\dim \pi}$ such that

- r_i, $0 \leqslant i < l_1$, acts by multiplication with μ_i on S^1,
- s_i, $0 \leqslant i < k_0 + l_1$, acts trivially on S^1 and
- t_i, $0 \leqslant i < l_2$, acts by multiplication with -1 on S^1.

Invoking Theorem 5.3.3, this suffices to finish the proof.

In what follows, we will apply Lemma 5.4.1 repeatedly. Replace the basis elements y_i, $0 \leqslant i < n_1$ by $\tilde{y}_i = y_i x_i$ for $0 \leqslant i < n_1$. Then $\tilde{y}_i$ acts on S^1 by multiplication with μ_i, $0 \leqslant i < n_1$. Recall that we assumed $l_1 \leqslant n_1$. Since the subgroups of S^1 generated by the eigenvalues of the complexifications of π and ρ agree, for every $0 \leqslant i < l_1$ there are elements $a_{i,1} \ldots, a_{i,\alpha} \in \mathbb{Z}$, $0 \leqslant j_{i,1}, \ldots, j_{i,\alpha} < n_1$ and $a_{i,0} \in \{0, 1\}$ such that

$$\mu_i = \lambda_{j_1}^{a_{i,1}} \cdots \lambda_{j_\alpha}^{a_{i,\alpha}} \cdot (-1)^{a_{i,0}} ,$$

where $a_{i,0} = 0$ if -1 is not an eigenvalue of π. Replacing x_i, $0 \leqslant i < l_1$ by

$$r_i = x_i \tilde{y}_i^{-1} \tilde{y}_{j_{i,1}}^{a_{i,1}} \cdots \tilde{y}_{j_{i,\alpha(i)}}^{a_{i,\alpha(i)}} \cdot z_1^{a_{i,0}} ,$$

we obtain a new basis of $\mathbb{F}_{\dim\pi}$ consisting of r_i $(0 \leqslant i < l_1)$, x_i $(l_1 \leqslant i < n_1)$, $\tilde{y}_i$ $(0 \leqslant i < n_1)$, z_i $(0 \leqslant i < n_2)$ and w_i $(0 \leqslant i < m_0)$.

We distinguish whether -1 in an eigenvalue of ρ or not. If -1 is no eigenvalue of ρ, we produce elements s_i $(0 \leqslant i < (n_1 - l_1) + n_1 + n_2 + m_0)$ acting trivially on S^1, where we put $n_1 - l_1 = 0$, if $l_1 = n_1 = \infty$. Replace x_i by $x_i\tilde{y}_i^{-1}$ for $l_1 \leqslant i < n_1$ and then multiply $\tilde{y}_i$, $0 \leqslant i < n_1$ and z_i, $0 \leqslant i < n_2$ from the right with words in r_i, $0 \leqslant i < l_1$ so as to obtain these new basis elements s_i $(0 \leqslant i < (n_1 - l_1) + n_1 + n_2 + m_0)$. Since $\dim\pi = 2n_1 + n_2 + m_0 = l_1 + (n_1 - l_1) + n_1 + n_2 + m_0$ and $l_2 = 0$, we found a basis r_i $(0 \leqslant i < l_1)$, s_i $(0 \leqslant i < l_1 + k_0)$ of $\mathbb{F}_{\dim\pi}$ acting on S^1 as desired. This finishes the proof in the case -1 is no eigenvalue of ρ.

Now assume that -1 is an eigenvalue of ρ. We distinguish three further cases.
Case $l_1 < n_1$: There are elements $a_1 \ldots, a_\alpha \in \mathbb{Z}$, $0 \leqslant i_1, \ldots, i_\alpha < n_1$ and $a_0 \in \{0, 1\}$ such that

$$-1 = \lambda_{i_1}^{a_1} \cdots \lambda_{i_\alpha}^{a_\alpha} \cdot (-1)^{a_0} \,,$$

where $a_0 = 0$ if -1 is not an eigenvalue of π. Replace x_{l_1+1} by

$$t_1 = x_{l_1+1}\tilde{y}_{l_1+1}^{-1}\tilde{y}_{i_1}^{a_1} \cdots \tilde{y}_{i_\alpha}^{a_\alpha} z_1^{a_0} \,.$$

Case $l_1 = n_1$ and -1 is an eigenvalue of π: Put $t_1 = z_1$.
Case $l_1 = n_1$ and -1 is no eigenvalue of π: Since $2n_1 + m_0 = 2l_1 + l_2 + k_0$, in this case, π has a trivial subrepresentation of dimension 1 or π is infinite dimensional. Hence, we may assume that $m \geqslant 1$, since all $y_i, 0 \leqslant i < n_1$ act trivially on S^1. There are elements $a_1 \ldots, a_\alpha \in \mathbb{Z}$, $0 \leqslant i_1, \ldots, i_\alpha < n_1$ such that

$$-1 = \lambda_{i_1}^{a_1} \cdots \lambda_{i_\alpha}^{a_\alpha} \,.$$

Put

$$t_1 = w_1\tilde{y}_{i_1}^{a_1} \cdots \tilde{y}_{i_\alpha}^{a_\alpha} \,.$$

In all three cases, we obtain a basis of $\mathbb{F}_{\dim\pi}$ with elements r_i $(0 \leqslant i < l_1)$, possibly t_1 and some other elements such that

- r_i, $0 \leqslant i < l_1$, acts by multiplication with μ_i on S^1,
- t_1 acts by multiplication with -1 on S^1 and
- all other elements of the basis act on S^1 by multiplication with some element in $G \subset \mathrm{S}^1$.

We can multiply the elements different from r_i, $(0 \leqslant i < l_1)$, and t_1 in the basis by some word in the letters r_i, $0 \leqslant i < l_1$ and t_1 in order to obtain a basis r_i $(0 \leqslant i < l_1)$, s_i $(0 \leqslant i < \dim\pi - l_1 - 1)$, t_1 or r_i $(0 \leqslant i < l_1)$, s_i $(0 \leqslant i < \dim\pi - l_1)$ where all elements s_i act trivially on S^1. We used the convention $\dim\pi - l_1 = \infty$, if $l_1 = \dim\pi = \infty$. If $l_1 + k_0 < \infty$, replace s_i, $(l_1 + k_0 \leqslant i < l_1 + k_0 + l_2 - 1)$ by $t_{i-k+2} = s_i \cdot t_1$, in order to obtain a basis r_i $(0 \leqslant i < l_1)$, s_i $(0 \leqslant i < l_1 + k_0)$, t_i $(0 \leqslant i < l_2)$ of $\mathbb{F}_{\dim\pi}$ acting on S^1 as desired. If $l_1 + k_0 = \infty$, then replace l_2-many s_i by $s_i t_1$ so as to obtain the new basis r_i $(0 \leqslant i < l_1)$, s_i $(0 \leqslant i < l_1 + k_0)$, t_i $(0 \leqslant i < l_2)$ of $\mathbb{F}_{\dim\pi}$ acting on S^1 as desired. This finished the proof. □

5.4.2 The classification of free Bogoliubov crossed products associated with periodic representations of the integers is equivalent to the isomorphism problem for free group factors

The classification of free Bogoliubov crossed products associated with non-faithful, that is periodic, orthogonal representations of $\mathbb{Z}$ implies a solution to the isomorphism problem for free group factors. For example, if $\mathbb{1}$ denotes the trivial orthogonal representation of $\mathbb{Z}$, we have $M_{n\cdot\mathbb{1}} \cong \mathrm{L}\mathbb{F}_n \overline{\otimes} \mathrm{L}\mathbb{Z}$. So, proving whether $M_{n\cdot\mathbb{1}} \cong M_{m\cdot\mathbb{1}}$ or not for different n and m amounts to solving the isomorphism problem for free group factors. More generally, we have the following result.

Theorem 5.4.3. *Let π be a periodic orthogonal representation of the integers. If π is trivial, then $A_\pi \subset M_\pi$ is isomorphic to an inclusion $1 \otimes \mathrm{L}^\infty([0,1]) \subset \mathrm{L}\mathbb{F}_{\dim\pi} \overline{\otimes} \mathrm{L}^\infty([0,1])$. If π is one dimensional and non-trivial, then $(A_\pi \subset M_\pi) \cong (\mathbb{C}^2 \otimes 1 \otimes \mathrm{L}^\infty([0,1]) \subset \mathrm{M}_2(\mathbb{C}) \otimes \mathrm{L}^\infty([0,1]) \otimes \mathrm{L}^\infty([0,1]))$. If π has dimension at least 2, let T be the index of the kernel of π in $\mathbb{Z}$. Then $(A_\pi \subset M_\pi) \cong (\mathbb{C}^T \otimes \mathrm{L}^\infty([0,1]) \subset \mathrm{L}\mathbb{F}_r \overline{\otimes} \mathrm{L}^\infty([0,1]))$, where $\mathrm{L}\mathbb{F}_r$ is an interpolated free group factor with parameter*

$$r = 1 + \frac{1}{T}(\dim\pi - 1)\,.$$

Proof. The case where π is trivial, follows directly from the definition of $\Gamma(H, \mathbb{Z}, \pi)''$. To prove all other cases, by Theorem 5.4.2, it suffices to consider representations $\pi = \pi_0 \oplus n\cdot\mathbb{1}$ with π_0 irreducible and non-trivial and $n \in \mathbb{N} \cup \{\infty\}$.

We first consider irreducible representations. The case of π one dimensional is immediately verified from the definition of $M_\pi = \Gamma(H, \mathbb{Z}, \pi)''$. If π has dimension 2 and is irreducible denote by $\lambda = e^{\frac{2\pi i}{T}}$ and $\overline{\lambda} = e^{-\frac{2\pi i}{T}}$, with

$T = [\mathbb{Z} : \ker \pi] \in \mathbb{N}_{\geqslant 2}$, the eigenvalues of $\pi_{\mathbb{C}}$. Then

$$\begin{aligned} M_\pi &\cong (\mathrm{L}\mathbb{Z} \overline{\otimes} \mathrm{L}\mathbb{Z}) *_{1 \otimes \mathrm{L}\mathbb{Z}} (\mathrm{L}\mathbb{Z} \rtimes_\lambda \mathbb{Z}) \\ &\cong (\mathrm{L}\mathbb{Z} \overline{\otimes} \mathbb{C}^T \overline{\otimes} \mathrm{L}^\infty([0,1])) *_{1 \otimes \mathbb{C}^T \overline{\otimes} \mathrm{L}^\infty([0,1])} (\mathrm{L}^\infty([0,1]) \overline{\otimes} \mathrm{M}_T(\mathbb{C}) \overline{\otimes} \mathrm{L}^\infty([0,1])) \\ &\cong ((\mathrm{L}\mathbb{Z} \overline{\otimes} \mathbb{C}^T) *_{1 \otimes \mathbb{C}^T} (\mathrm{L}^\infty([0,1]) \overline{\otimes} \mathrm{M}_T(\mathbb{C}))) \overline{\otimes} \mathrm{L}^\infty([0,1]) \,. \end{aligned}$$

Since $(\mathrm{L}\mathbb{Z} \overline{\otimes} \mathbb{C}^T) *_{1 \otimes \mathbb{C}^T} (\mathrm{L}^\infty([0,1]) \overline{\otimes} \mathrm{M}_T(\mathbb{C}))$ is a non-amenable factor by Theorem 5.2.5, Theorem 5.2.4 shows that

$$((\mathrm{L}\mathbb{Z} \overline{\otimes} \mathbb{C}^T) *_{1 \otimes \mathbb{C}^T} (\mathrm{L}^\infty([0,1]) \overline{\otimes} \mathrm{M}_T(\mathbb{C}))) \cong \mathrm{L}\mathbb{F}_r$$

with

$$r = 1 + 1 - (1 - \frac{1}{T}) = 1 + \frac{1}{T}(\dim \pi - 1) \,.$$

Moreover,

$$\begin{aligned} (A_\pi \subset M_\pi) &\cong (1 \otimes \mathbb{C}^T \overline{\otimes} \mathrm{L}^\infty([0,1]) \subset ((\mathrm{L}\mathbb{Z} \overline{\otimes} \mathbb{C}^T) *_{1 \otimes \mathbb{C}^T} \\ &\qquad (\mathrm{L}^\infty([0,1]) \overline{\otimes} \mathrm{M}_T(\mathbb{C}))) \overline{\otimes} \mathrm{L}^\infty([0,1])) \\ &\cong (\mathbb{C}^T \otimes \mathrm{L}^\infty([0,1]) \subset \mathrm{L}\mathbb{F}_r \overline{\otimes} \mathrm{L}^\infty([0,1])) \,. \end{aligned}$$

Consider now $\pi = \pi_0 \oplus n \cdot \mathbb{1}$ for an irreducible, non-trivial and non-faithful representation of dimension two π_0. The case where π_0 is of dimension one and has eigenvalue -1 is similar, but simpler. Let $T = [\mathbb{Z} : \ker \pi_0] \in \mathbb{N}_{\geqslant 2}$ and $n \in \mathbb{N} \cup \{\infty\}$. Let $r_0 = 1 + \frac{1}{T}$. Then Theorems 5.2.4 and 5.2.5 imply that

$$\begin{aligned} M_{\pi_0 \oplus n \cdot \tau} &\cong (\mathrm{L}\mathbb{F}_n \overline{\otimes} \mathrm{L}\mathbb{Z}) *_{1 \otimes \mathrm{L}\mathbb{Z} \cong \mathbb{C}^T \otimes \mathrm{L}^\infty([0,1])} (\mathrm{L}\mathbb{F}_{r_0} \overline{\otimes} \mathrm{L}^\infty([0,1])) \\ &\cong (\mathrm{L}\mathbb{F}_n \otimes \mathbb{C}^T *_{1 \otimes \mathbb{C}^T} \mathrm{L}\mathbb{F}_{r_0}) \overline{\otimes} \mathrm{L}^\infty([0,1]) \\ &\cong \mathrm{L}\mathbb{F}_r \otimes \mathrm{L}^\infty([0,1]) \,, \end{aligned}$$

with

$$r = 1 + \frac{1}{T}(n-1) + r_0 - (1 - \frac{1}{T}) = \frac{1}{T}(\dim(\pi_0 \oplus n \cdot \mathbb{1}) - 1) \,.$$

Also

$$(A_\pi \subset M_\pi) \cong (\mathbb{C}^T \otimes \mathrm{L}^\infty([0,1]) \subset \mathrm{L}\mathbb{F}_r \overline{\otimes} \mathrm{L}^\infty([0,1]))$$

and this finishes the proof. □

5.4.3 A flexibility result for representations with one pair of non-trivial eigenvalue

In this section, we will show that all free Bogoliubov crossed products associated with almost periodic orthogonal representations of $\mathbb{Z}$ with a single non-trivial irreducible component, which is faithful, are isomorphic.

Proposition 5.4.4. *Let π_i for $i \in \{1,2\}$ be almost periodic orthogonal representations of $\mathbb{Z}$ having the same dimension. Assume that their complexifications $(\pi_i)_\mathbb{C}$ each have a single pair of non-trivial eigenvalues $\lambda_i, \overline{\lambda_i} \in e^{2\pi i \mathbb{R}\setminus\mathbb{Q}}$ with any multiplicity. Then $M_{\pi_1} \cong M_{\pi_2}$ by an isomorphism, which carries A_{π_1} onto A_{π_2}.*

Proof. By Theorem 5.4.2 is suffices to consider the case where the eigenvalue λ_i of $(\pi_i)_\mathbb{C}$ has multiplicity one. Theorem 5.3.3 shows that

$$M_{\pi_1} \cong (\mathrm{L}\mathbb{F}_{\dim \pi_1 - 1} \otimes \mathrm{L}^\infty(\mathrm{S}^1)) *_{1\otimes \mathrm{L}^\infty(\mathrm{S}^1)} (\mathbb{Z} \ltimes_{\lambda_i} \mathrm{L}^\infty(\mathrm{S}^1)),$$

by an isomorphism, which caries A_{π_i} onto $\mathrm{L}^\infty(\mathrm{S}^1)$. Taking an orbit equivalence of the ergodic hyperfinite II_1 equivalence relations induced by $\mathbb{Z} \overset{\lambda_1}{\curvearrowright} \mathrm{S}^1$ and $\mathbb{Z} \overset{\lambda_2}{\curvearrowright} \mathrm{S}^1$, we obtain an isomorphism $\mathbb{Z} \ltimes_{\lambda_1} \mathrm{L}^\infty(\mathrm{S}^1) \cong \mathbb{Z} \ltimes_{\lambda_2} \mathrm{L}^\infty(\mathrm{S}^1)$, which preserves $\mathrm{L}^\infty(\mathrm{S}^1)$ globally. This can be extended to an isomorphism $M_{\pi_1} \cong M_{\pi_2}$, which carries A_{π_1} onto A_{π_2}. □

Corollary 5.4.5. *All faithful two dimensional representations of $\mathbb{Z}$ give rise to isomorphic free Bogoliubov crossed products.*

5.4.4 Some remarks on a possible classification of Bogoliubov crossed products associated with almost periodic orthogonal representations

In Theorem 5.4.2 we showed that the isomorphism class of free Bogoliubov crossed products associated with almost periodic orthogonal representations of $\mathbb{Z}$ depends at most on the concrete subgroup of S^1 generated by the eigenvalues of its complexification. However, Theorem 5.4.3 and Proposition 5.4.4 both show that there are orthogonal representations π, ρ of $\mathbb{Z}$ such that these subgroups of S^1 are not equal and still they give rise to isomorphic free Bogoliubov crossed products. This answers a question of Shlyakhtenko, asking whether a complete invariant for the isomorphism class of the free Bogoliubov crossed products associated with an orthogonal representation π of $\mathbb{Z}$ is $\bigoplus_{n\geqslant 1}\pi^{\otimes n}$ up to amplification. By Theorem 5.4.3, the classification of free Bogoliubov crossed products associated with non-faithful orthogonal representations of $\mathbb{Z}$ is equivalent to the isomorphism problem for free group factors. However, assuming that M_π is a factor, i.e. that π is faithful, the abstract isomorphism class of the group generated by the eigenvalues of the complexification of π could be an invariant. Due to the fact that the isomorphisms found in Theorem 5.4.3 preserve the subalgebra $A_\pi \subset M_\pi$ for non-faithful orthogonal representations, we believe that this abstract group is indeed an invariant for infinite dimensional representations.

Conjecture 5.4.6. *The abstract isomorphism class of the subgroup generated by the eigenvalues of the complexification of an infinite dimensional faithful almost periodic orthogonal representation of $\mathbb{Z}$ is a complete invariant for isomorphism of the associated free Bogoliubov crossed product.*

5.5 Solidity and strong solidity for free Bogoliubov crossed products

The proof of the following result can be extracted literally from the proof of [195, Theorem 1]. It shows that the dimension of the almost periodic part of an orthogonal representation of $\mathbb{Z}$ is relevant for the isomorphism class of its free Bogoliubov crossed product. We give a full prove for the convenience of the reader. Recall that we denote by $\mathbb{1}$ the trivial orthogonal representation of the integers.

Theorem 5.5.1. *Let π be a one dimensional orthogonal representation of $\mathbb{Z}$. Then the free Bogoliubov crossed products M_λ and $M_{\lambda \oplus \pi}$ are isomorphic to* $\mathrm{L}\mathbb{F}_2$.

Proof. First note that π is either the trivial representation or $\pi(1)$ acts via multiplication with -1 on $\mathbb{R}$. We treat both cases simultaneously. We have $M_\lambda \cong \mathrm{L}(\mathbb{F}_\infty) \rtimes \mathbb{Z}$, where $\mathbb{Z} \curvearrowright \mathbb{F}_\infty$ by shifting a free basis $(g_n)_{n \in \mathbb{Z}}$. Denote by u the natural generator of $\mathbb{Z}$ in the copy $\mathrm{L}(\mathbb{Z}) \subset \mathrm{L}(\mathbb{F}_\infty) \rtimes \mathbb{Z}$ and denote by v the generator of $\mathbb{Z}$ in $\mathrm{L}(\mathbb{Z}) \otimes 1 \subset \mathrm{L}(\mathbb{Z}) \rtimes_{\pm 1} \mathbb{Z}$.

We claim that $\{u, v\}$ and $\{u_{g_0}\}$ generate free subalgebras inside $M_{\lambda \oplus \pi}$. Since elements in $\{u^k v^l \mid k, l \in \mathbb{Z}\}$ span $\mathrm{L}(\mathbb{Z}) \rtimes_{\pm 1} \mathbb{Z}$ densely, it suffices to check that alternating words w in $\{u^k v^l \mid k, l \in \mathbb{Z}, (k, l) \neq (0, 0)\}$ and $\{u_{g_0}^m \mid m \in \mathbb{Z}^\times\}$ satisfy $\tau(w) = 0$. Take such a word w. Because of the commutation relation $v u_{g_n} = u_{g_{n+1}} v$ in $\mathrm{L}(\mathbb{F}_\infty) \rtimes \mathbb{Z}$ and $vu = \pm uv$ in $\mathrm{L}(\mathbb{Z}) \rtimes_{\pm 1} \mathbb{Z}$, we can rewrite $w = \varepsilon \cdot w' \cdot v^k$, where

- $\varepsilon \in \{1, -1\}$,

- w' an alternating word in $\{u_{g_n}^l \mid n \in \mathbb{Z}, l \in \mathbb{Z}^\times\}$ and $\{u^m \mid m \in \mathbb{Z}^\times\}$ and
- $k \in \mathbb{Z}$ such that $k \neq 0$ if w' is trivial.

Denote by $\mathrm{E} : M_{\lambda\oplus\pi} \to \mathrm{L}(\mathbb{Z})$ the natural conditional expectation of the free product decomposition $M_{\lambda\oplus\pi} \cong M_\lambda *_{\mathrm{L}(\mathbb{Z})} M_\pi$. Since $\mathrm{E}(u_{g_n}^l) = 0 = \mathrm{E}(u^m)$ for all $n \in \mathbb{Z}$ and all $l, m \in \mathbb{Z}^\times$, we obtain

$$\tau(w) = \varepsilon\tau(w'v^k) = \varepsilon\tau(\mathrm{E}(w')\mathrm{E}(v^k)) = \varepsilon\tau(w')\tau(v^k) = 0\,,$$

where the last equality steams from the fact that $k \neq 0$ if $w' = 1$. Since $\{u, v, u_{g_0}\}$ generates $M_{\lambda\oplus\pi}$ as a von Neumann algebra, it follows that

$$M_{\lambda\oplus\pi} \cong \{u, v\}'' * \{u_{g_0}\}'' \cong (\mathrm{L}(\mathbb{Z}) \rtimes_{\pm 1} \mathbb{Z}) * \mathrm{L}(\mathbb{Z}) \cong \mathrm{L}(\mathbb{F}_2)$$

by Theorems 5.2.4 and 5.2.5. □

The fact that the left regular representation plus a trivial one dimensional representation gives rise to a strongly solid free Bogoliubov crossed product, triggered the following observation.

Theorem 5.5.2. *Let π be an orthogonal representation of $\mathbb{Z}$ that is the direct sum of a mixing representation and a representation of dimension at most one. Then M_π is strongly solid.*

This theorem follows from the next, more general, one. Its proof can be taken almost literally from [114, Theorem 1.8]. We include a proof for the convenience of the reader.

Theorem 5.5.3. *Let $A \subset N$ be a mixing inclusion of A into a strongly solid, non-amenable, tracial von Neumann algebra. Let $A \subset B$ an inclusion of A into an amenable, tracial von Neumann algebra. Then $M = N *_A B$ is strongly solid.*

Proof. We first show that $B \subset M$ is mixing. As in [114, Theorem 1.8], we have to show that for every sequence $(b_n)_n$ in $(B)_1$ with $b_n \to 0$ weakly and for all $a, b \in B$, $x, y \in N \ominus A$ we have

$$\mathrm{E}_A(x\mathrm{E}_A(ab_nb)y) \xrightarrow{\|\ \|_2} 0\,.$$

Since $b_n \to 0$ weakly, also $\mathrm{E}_A(ab_nb) \to 0$ weakly. The fact that $A \subset N$ is mixing, then implies that $\|\mathrm{E}_A(x\mathrm{E}_A(ab_nb)y)\|_2 \to 0$.

Let $Q \subset M$ be a diffuse, amenable von Neumann subalgebra and write $P = \mathcal{N}_M(Q)''$. Let $p \in \mathcal{Z}(P)$ be the maximal projection such that Pp has no amenable direct summand. We assume $p \neq 0$ and deduce a contradiction. Let ω be a non-principal ultrafilter. By Theorem 5.2.11 we have $p = e + f$ with $e, f \in \mathcal{Z}((Pp)' \cap pMp) \cap \mathcal{Z}((Pp)' \cap (pMp)^\omega)$ such that

- $e((Pp)' \cap (pMp)^\omega) = e((Pp)' \cap pMp)$ and this algebra is atomic and
- $f((Pp)' \cap (pMp)^\omega)$ is diffuse.

Either $e \neq 0$ or $f \neq 0$. In both cases, we will deduce that $Pp \prec_M N$.

If $e \neq 0$ let $e_0 \in (Pp)' \cap pMp$ be a minimal projection. Then $(Pe_0)' \cap (e_0 M e_0)^\omega = \mathbb{C}e_0$, so Theorem 5.2.13 applies to $Ae_0 \subset e_0 M e_0$ and $Pe_0 \subset \mathcal{N}_{e_0 M e_0}(Qe_0)''$. We obtain that one of the following holds.

- $Qe_0 \prec_M A$,
- $Pe_0 \prec_M N$,
- $Pe_0 \prec_M B$ or
- Pe_0 is amenable relative to A.

The first item implies that $Qe_0 \prec B$ and since $B \subset M$ is mixing, Lemma 5.2.12 shows that $Pe_0 \prec B$. So the first and the last two items imply that Pe_0 has an amenable direct summand, which contradicts the choice of p. We obtain $Pp \prec_M N$ in the case $e \neq 0$.

If $f \neq 0$ then Theorem 5.2.10 applied to $Pf \subset fMf$ shows that one of the following holds.

- $(Pf)' \cap (fMf)^\omega \prec_{M^\omega} A^\omega$,
- $Pf \prec_M N$,
- $Pf \prec_M B$ or
- there is a non-zero projection $f_0 \in \mathcal{Z}((Pf)' \cap fMf)$ such that Pf_0 is amenable relative to A.

The first item implies $(Pf)' \cap (fMf)^\omega \prec_{M^\omega} B^\omega$ and since $B \subset M$ is mixing, Theorem 5.2.14 shows that $Pf \prec_M B$. So the first and the last two items imply that Pf has an amenable direct summand, contradicting the choice of p. This shows $Pp \prec_M N$ in the case $f \neq 0$.

We showed $Pp \prec_M N$. Let $p_0 \in P$, $q \in Q$, $p_0 \leqslant p$ be non-zero projections, $v \in pMq$ satisfying $vv^* = p_0$ and $\phi : p_0 P p_0 \to qNq$ a *-homomorphism such that $xv = v\phi(x)$ for all $x \in p_0 P p_0$. We have $v^*v \in \phi(p_0 P p_0)' \cap M$. Since $p_0 P p_0$ has no amenable direct summand it follows that $\phi(p_0 P p_0) \nprec_M A$, and hence Theorem 5.2.9 shows that $v^*v \in N$. So we can conjugate P by a unitary in order to assume $p_0 P p_0 \subset N$. Take partial isometries $w_1, \ldots, w_n \in P$ such that $z = \sum_i w_i w_i^* \in \mathcal{Z}(P)$ and $w_i^* w_i = \tilde{p} \leqslant p_0$ for all $i = 1, \ldots, n$. Then we obtain a *-homomorphism

$$\psi : Pz \to \mathrm{M}_n(\mathbb{C}) \otimes \tilde{p} N \tilde{p} : x \mapsto (w_i^* x w_j)_{i,j} .$$

By [107, Proposition 5.2], know that $\mathrm{M}_n(\mathbb{C}) \otimes \tilde{p} N \tilde{p}$ is strongly solid. This contradicts

$$\psi(Pz) \subset \mathcal{N}_{\mathrm{M}_n(\mathbb{C}) \otimes \tilde{p} N \tilde{p}}(\psi(Az))''$$

and the choice of p. □

Proof of Theorem 5.5.2. Write $\pi = \pi_1 \oplus \pi_2$ with π_1 mixing and $\dim \pi_2 \leqslant 1$. Then $M_\pi \cong M_{\pi_1} *_A M_{\pi_2}$. Since $A \subset M_{\pi_1}$ is mixing by [214, Theorem D.4], it is strongly solid by [112, Theorem B]. Also M_{π_2} is amenable, so Theorem 5.5.3 applies. □

We have a partial converse to the previous theorem.

Theorem 5.5.4. *Let π be an orthogonal representation of $\mathbb{Z}$ with a rigid subspace of dimension at least two. Then M_π is not solid.*

Proof. Let ω be a non-principal ultrafilter. Let $\xi, \eta \in H$ be orthogonal vectors such that there is a sequence $(n_k)_k$ going to infinity in $\mathbb{Z}$ and $\pi(n_k)\xi \to \xi$, $\pi(n_k)\eta \to \eta$ if $k \to \infty$. Then $[u_{n_k}] \in A^\omega$ is a Haar unitary and hence $P = \{s(\xi), s(\eta)\}''$ is a non-amenable subalgebra such that $P' \cap A^\omega \subset P' \cap M_\pi^\omega$ is diffuse. Applying [153, Proposition 7] to $P \subset M_\pi$ shows that M_π is not solid. □

We conjecture that the previous theorem is sharp.

Conjecture 5.5.5. *Let π be an orthogonal representation of $\mathbb{Z}$. Then the following are equivalent.*

- *M_π is strongly solid.*
- *M_π is solid.*
- *π has no rigid subspace of dimension two.*

The Theorems 5.5.2 and 5.5.4 of this work as well as Theorem A of [106] on free Bogoliubov crossed products that do not have property Gamma are supporting evidence for our conjecture. We explain how Houdayer's result is related it.

Theorem 5.5.6 (See Theorem A of [106]). *Let G be a countable discrete group and $\pi : G \to \mathcal{O}(H)$ any faithful orthogonal representation such that* $\dim H \geqslant 2$ *and $\pi(G)$ is discrete in $\mathcal{O}(H)$ with respect to the strong topology. Then $\Gamma(H)'' \rtimes_\pi G$ is a II_1 factor which does not have property Gamma.*

First of all, note that in view of Proposition 7 of [153], being non-Gamma can be considered as a weak form of solidity. Secondly, we remark that an orthogonal representation $\pi : G \to \mathcal{O}(H)$ has discrete range, if and only if the whole Hilbert space H is not rigid in our terminology. This explains the link between our conjecture and the result of Houdayer.

5.6 Rigidity results

In this section, we want to show how to extract some information about π from the von Neumann algebra M_π. As an application, we exhibit orthogonal representations of $\mathbb{Z}$ that cannot give rise to isomorphic free Bogoliubov crossed products.

Theorem 5.6.1. *Let π_1, π_2 be orthogonal representations of $\mathbb{Z}$ such that each of them has a finite dimensional invariant subspace of dimension 2. Assume that $M = M_{\pi_1} \cong M_{\pi_2}$. Let $A = A_{\pi_1}$ and identify A_{π_2} with a subalgebra $B \subset M$. Then there is a finite index A-B-subbimodule of $\mathrm{L}^2(M)$.*

Proof. We want to use Theorem 5.2.8 in order to find a finite index A-B bimodule in $\mathrm{L}^2(M)$. So we have to verify its assumptions. Corollary 5.3.9 implies that the normalisers of A and B are non-amenable. So by Corollary 5.2.7, $A \prec_M^{\mathrm{f}} B$ and $B \prec_M^{\mathrm{f}} A$ hold. By Proposition 5.3.8, every right finite A-A subbimodule of $\mathrm{L}^2(M)$ lies in $\mathrm{L}^2(\mathrm{QN}_M(A)'')$. So Theorem 5.2.8 says that there is a finite index A-B-subbimodule of $\mathrm{L}^2(M)$. □

Corollary 5.6.2. *Let π_1, π_2 be two orthogonal representations of $\mathbb{Z}$ having a finite dimensional subrepresentation of dimension at least 2. Let $A_1 \subset M_1$ and $A_2 \subset M_2$ be the inclusions of the free Bogoliubov crossed products associated with π_1 and π_2, respectively. Assume that $M_1 \cong M_2$. Then there are projections $p_1 \in A_1$, $p_2 \in A_2$ and an isomorphism $\phi : A_1p_1 \to A_2p_2$ preserving the normalised traces such that the bimodules ${}_{A_1p_1}(p_1\mathrm{L}^2(M)p_1)_{A_1p_1}$ and ${}_{\phi(A_1p_1)}(p_2\mathrm{L}^2(M)p_2)_{\phi(A_1p_1)}$ are isomorphic.*

Proof. By Theorem 5.6.1, there are projections $p_1 \in A_1$, $p_2 \in A_2$, an isomorphism $\phi : A_1p_1 \to A_2p_2$ and a partial isometry $v \in p_1Mp_2$ such that $av = v\phi(a)$ for all $a \in A_1p_1$. Denote by q_1 and q_2 the left and right support of v, respectively. Cutting down p_1 and p_2, we can assume that $\operatorname{supp} \mathrm{E}_{A_1}(q_1) = p_1$ and $\operatorname{supp} \mathrm{E}_{A_2}(q_2) = p_2$. The bimodules ${}_{A_1p_1}(q_1\mathrm{L}^2(M)q_1)_{A_1p_1}$ and ${}_{\phi(A_2p_2)}(q_2\mathrm{L}^2(M)q_2)_{\phi(A_2p_2)}$ are isomorphic.

Since p_1 is the central support of q_1 in $A' \cap M$, there are projections $e_n \in A$, $n \in \mathbb{N}$ such that $q_1 = \sum_n e_n$ and partial isometries $v_n^k \in A' \cap M$, $n \in \mathbb{N}$, $k \leqslant n$ such that $\sum_k v_n^k (v_n^k)^* = e_n$ and $(v_n^1)^* v_n^1 = e_n q_1$, $(v_n^k)^* v_n^k \leqslant q_1$, for all n and all $2 \leqslant k \leqslant n$. Since the multiplicity function of ${}_{A_1}\mathrm{L}^2(M)_{A_1}$ is constantly equal to infinity by Proposition 5.2.2, we find that

$$ {}_{Ae_n}(e_n q_1 \mathrm{L}^2(M) e_n q_1)_{Ae_n} \cong \bigoplus_{k \leqslant n} {}_{Ae_n}(v_n^k \mathrm{L}^2(M)(v_n^k)^*)_{Ae_n} \cong {}_{Ae_n}(e_n \mathrm{L}^2(M) e_n)_{Ae_n} \,, $$

for all n. So also

$$ {}_{Ap_1}(p_1 \mathrm{L}^2(M) p_1)_{Ap_1} \cong {}_{Ap_1}(q_1 \mathrm{L}^2(M) q_1)_{Ap_1} \,. $$

Similarly, we have ${}_{A_2p_2}(p_2\mathrm{L}^2(M)p_2)_{A_2p_2} \cong {}_{A_2p_2}(q_2\mathrm{L}^2(M)q_2)_{A_2p_2}$. This finishes the proof. □

A measure theoretic reformulation of Corollary 5.6.2 can be given as follows.

Corollary 5.6.3. *Let $(\mu_1, N_1), (\mu_2, N_2)$ be symmetric probability measures with multiplicity function on S^1 such that both have at least 2 atoms when counted with multiplicity. For $i = 1, 2$, let π_i be the orthogonal representation of $\mathbb{Z}$ by multiplication with $\mathrm{id}_{\mathrm{S}^1}$ on $\mathrm{L}^2{}_{\mathbb{R}}(\mathrm{S}^1, \mu_i, N_i)$. If $M_{\pi_1} \cong M_{\pi_2}$, then there are Lebesgue non-negligible Borel subsets $B_1, B_2 \subset \mathrm{S}^1$ and a Borel isomorphism $\varphi : B_1 \to B_2$ preserving the normalised Lebesgue measures such that*

$$ \varphi_* \left([\sum_{n \geqslant 0} \mu_1^{*n} * \delta_{\varphi(s)}]|_{B_1} \right) = [\sum_{n \geqslant 0} \mu_2^{*n} * \delta_s]|_{B_2} \,. $$

for Lebesgue almost every $s \in B_2$

Proof. Write $M = M_{\pi_1} \cong M_{\pi_2}$ and A_i, for $i \in \{1, 2\}$. Denote by $[\nu_i] = \int [\sum_{n \geqslant 0} \mu_i^{*n} * \delta_s] \, \mathrm{d}\lambda(s)$ the maximal spectral type of ${}_{A_i}\mathrm{L}^2(M)_{A_i}$ according to Proposition 5.2.3. By Corollary 5.6.2, there are projections $p_1 \in A_1$, $p_2 \in A_2$ and an isomorphism $\phi : A_1p_1 \to A_2p_2$ such that the bimodules ${}_{A_1p_1}(p_1\mathrm{L}^2(M)p_1)_{A_1p_1}$ and ${}_{\phi(A_1p_1)}(p_2\mathrm{L}^2(M)p_2)_{\phi(A_1p_1)}$ are isomorphic. The projections p_i are indicator functions of Lebesgue non-negligible Borel sets $B_i \subset \mathrm{S}^1$ and the isomorphism ϕ equals φ_* for some Borel isomorphism $\varphi : B_1 \to B_2$ preserving the normalised Lebesgue measures. Since the bimodules ${}_{A_1p_1}(p_1\mathrm{L}^2(M)p_1)_{A_1p_1}$ and ${}_{A_2p_2}(p_2\mathrm{L}^2(M)p_2)_{A_2p_2}$ are isomorphic via ϕ, their maximal spectral types are isomorphic via $\varphi \times \varphi$. Using their integral decomposition with respect to the projection on the first component of $\mathrm{S}^1 \times \mathrm{S}^1$ as it is calculated in Proposition 5.2.3, we obtain

$$\left(\int_{B_2} [\sum_{n\geqslant 0} \mu_2^{*n} * \delta_s]|_{B_2} \,\mathrm{d}\lambda(s)\right) = (\varphi \times \varphi)_* \left(\int_{B_1} [\sum_{n\geqslant 0} \mu_1^{*n} * \delta_s]|_{B_1} \,\mathrm{d}\lambda(s)\right)$$

$$= (\varphi \times \mathrm{id})_* \left(\int_{B_2} [\sum_{n\geqslant 0} \mu_1^{*n} * \delta_{\varphi(s)}]|_{B_1} \,\mathrm{d}\lambda(s)\right)$$

$$= \left(\int_{B_2} \varphi_*([\sum_{n\geqslant 0} \mu_1^{*n} * \delta_{\varphi(s)}]|_{B_1}) \,\mathrm{d}\lambda(s)\right)$$

As a result, for almost every $s \in B_2$, we obtain the equality

$$\varphi_* \left([\sum_{n\geqslant 0} \mu_1^{*n} * \delta_{\varphi(s)}]|_{B_1}\right) = [\sum_{n\geqslant 0} \mu_2^{*n} * \delta_s]|_{B_2}\,.$$

□

The next theorem follows by applying the previous one to some special cases.

Theorem 5.6.4. *No free Bogoliubov crossed product associated with a representation in the following classes is isomorphic to a free Bogoliubov crossed product associated with a representation in the other classes.*

1. *The class of representations $\lambda \oplus \pi_{\mathrm{ap}}$, where λ is a multiple of the left regular representation of $\mathbb{Z}$ and π_{ap} is a faithful almost periodic representation of dimension at least 2.*

2. *The class of representations $\lambda \oplus \pi_{\mathrm{ap}}$, where λ is a multiple of the left regular representation of $\mathbb{Z}$ and π_{ap} is a non-faithful almost periodic representation of dimension at least 2.*

3. *The class of representations $\rho \oplus \pi_{\mathrm{ap}}$, where ρ is a representations of $\mathbb{Z}$ by multiplication with $\mathrm{id}_{\mathrm{S}^1}$ on $\mathrm{L}^2{}_{\mathbb{R}}(\mathrm{S}^1, \mu)$, μ is a probability measure on S^1 such that μ^{*n} is singular for all n and π_{ap} is a faithful almost periodic representation of dimension at least 2.*

4. *The class of representations $\rho \oplus \pi_{\mathrm{ap}}$, where ρ is a representations of $\mathbb{Z}$ by multiplication with $\mathrm{id}_{\mathrm{S}^1}$ on $\mathrm{L}^2{}_{\mathbb{R}}(\mathrm{S}^1, \mu)$, μ is a probability measure on S^1 such that μ^{*n} is singular for all n and π_{ap} is a non-faithful almost periodic representation of dimension at least 2.*

5. *Faithful almost periodic representations of dimension at least* 2*.*

6. *Non-faithful, almost periodic representations of dimension at least* 2*.*

7. *The class of representations* $\rho \oplus \pi$*, where* ρ *is mixing and* $\dim \pi \leqslant 1$*.*

Note that by [112], there are measures as mentioned item (iii) and (iv).

Proof. By Theorem 5.5.3, all free Bogoliubov crossed products associated with representations in 7 are strongly solid, but for all other free Bogoliubov crossed products $A \subset M$ is an amenable diffuse von Neumann subalgebra with a non-amenable normaliser.

It remains to consider representations in (i) to (vi). They satisfy the requirements of Corollaries 5.6.2 and 5.6.3.

We first claim that representations from (i) to (vi) with a faithful and non-faithful almost periodic part, respectively, cannot give rise to isomorphic free Bogoliubov crossed products. Let π be an orthogonal representation of $\mathbb{Z}$ and let $B \subset \mathrm{S}^1$ be Lebesgue non-negligible. The subgroup generated by the eigenvalues of the complexification of π is dense if and only if the almost periodic part of π is faithful. So by Section 5.2.4, the atoms of the spectral invariant of ${}_{pA_\pi} p\mathrm{L}^2(M)p_{pA_\pi}$ are an ergodic equivalence relation on $B \times B$ if and only if π has a faithful almost periodic part. So Corollary 5.6.2 proves our claim.

Let us now consider the weakly mixing part of the representations in the theorem. It is known that the spectral measure of the left regular representation of $\mathbb{Z}$ on $\ell^2_{\mathbb{R}}(\mathbb{Z})$ is the Lebesgue measure. So from Corollary 5.6.3, it follows that the representations whose weakly mixing part is the left regular representation, cannot give a free Bogoliubov crossed product isomorphic to a free Bogoliubov crossed product associated with any of the other representations in the theorem. Finally, note that for any non-zero projection $p \in A_\pi$ the bimodules ${}_{pA_\pi}\mathrm{L}^2(pM_\pi p)_{pA_\pi}$ is a direct sum of finite index bimodules if and only if the representation π has no weakly mixing part. So appealing to Corollary 5.6.2, we finish the proof. ☐

Chapter 6

A Connection between easy quantum groups, varieties of groups and reflection groups

This chapter is based on our joint work with Moritz Weber [184]. We prove that a fairly large class of compact quantum groups injects into the lattice of reflection groups via a natural construction. More precisely, we associate with certain easy quantum groups $\mathbb{G}$, in the sense of Banica and Speicher, a normal subgroup of the infinite free product $\mathbb{Z}_2^{*\infty}$ of the cyclic group of order two, which completely remembers the compact quantum group $\mathbb{G}$. Exploiting this relation, we use the theory of varieties of groups in order to show that easy quantum groups are not classifiable. Furthermore, we construct an inverse to the above map, which associates, by means of a quantum isometry group construction, an easy quantum group with certain reflection groups. This gives rise to a large number of new quantum isometry groups.

Introduction

In Connes' noncommutative geometry [53], the correct replacement for compact groups is given by Woronowicz's compact quantum groups [233, 236]. They are established due to a definition by a natural set of axioms, the natural development of a structural theory and a Tannaka-Krein type result identifying their categories of representations precisely as the concrete compact tensor

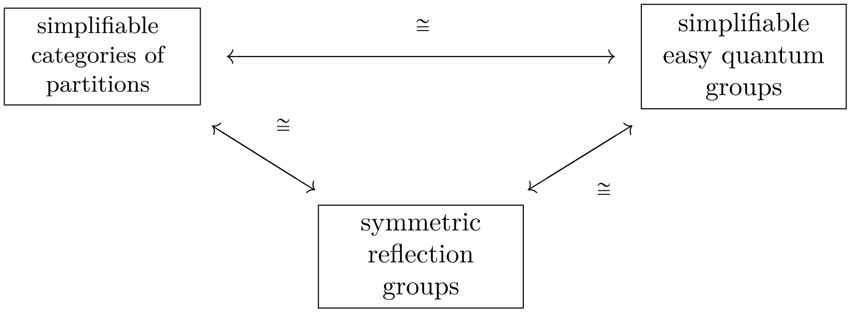
C*-categories [234]. One of the most intriguing questions in noncommutative geometry is the question for classification of noncommutative objects in terms of classical data.

The main result of this chapter states that there is a lattice isomorphism between specific classes of compact quantum groups and reflection groups. This opens new perspectives for research on compact quantum groups, since, as we show, results and techniques from the theory of reflection groups and varieties of groups (see Section 6.1.4) now can be applied to compact quantum groups.

There are three primary sources of compact quantum groups. Next to q-deformations of compact Lie groups [122, 68, 186] and quantum isometry groups [102] the third important class of compact quantum groups consists of Banica's and Speicher's *easy quantum groups* [24]. Easy quantum groups are defined by a combinatorial condition on their categories of representations, which naturally connects them to Voiculescu's free probability theory in Speicher's combinatorial setting. We explain this in detail later.

Our results give a concluding answer to the question whether the classification of easy quantum groups is feasible. Embedding the lattice of *varieties of groups* into the lattice of easy quantum groups, we show that a complete classification of easy quantum groups is impossible. This fact gives a new direction to the research in this field of mathematics by emphasising the need for global structural results on the lattice of easy quantum groups and the need for search of the most useful subclasses of easy quantum groups. We give results in both directions, making use of new techniques that are available because of our work.

The combinatorial description, which categories of representations of easy quantum groups satisfy by definition, goes by the name of *categories of partitions*. We introduce the natural condition of a simplifiable easy quantum group and the notion of a symmetric reflection group. Using these notations, our main result can be stated as a commuting diagram of lattice isomorphisms and anti-isomorphisms, respectively:



We give an explicit description of all arrows in this diagram in Section 6.6.1.

As mentioned before, *quantum isometry groups* are another source of compact quantum groups. They are particularly interesting, as they constitute the natural replacement for isometry groups of manifolds in Connes' noncommutative geometry. For example, they were calculated in the context of the standard model of particle physics [31]. In general, however, the calculation of quantum isometry groups poses a notoriously difficult problem. We describe the isomorphism of lattices between simplifiable easy quantum and symmetric reflection groups in terms of a quantum isometry group construction. This gives, on the one hand, an explicit method for passing from a symmetric reflection group to its associated easy quantum group. On the other hand, it solves the problem of calculating the quantum isometry groups of a fairly large number of quantum spaces, namely of the group C*-algebras of symmetric reflection groups.

We now put the definition of easy quantum groups and our result in a historical context. Let G be a (classical) Lie group and consider the C*-algebra $\mathrm{C}(G)$ of continuous functions on G. By means of the fundamental representation $u \in \mathrm{C}(G) \otimes \mathrm{M}_n(\mathbb{C})$, we can view $\mathrm{C}(G)$ as a universal C*-algebra:

$$\mathrm{C}(G) \cong \mathrm{C}^*\big(u_{ij}, 1 \leqslant i,j \leqslant n \,|\, \text{the matrices } (u_{ij}) \text{ and } (u_{ij}^*) \text{ are unitaries}, u_{ij}u_{kl} = u_{kl}u_{ij}, (R_G)\big),$$

where (R_G) are some further relations of the generators u_{ij}. The liberation G^+ of G is a compact quantum group given by the universal C*-algebra

$$\mathrm{C}(G^+) = \mathrm{C}^*\big(u_{ij}, 1 \leqslant i,j \leqslant n \,|\, \text{the matrices } (u_{ij}) \text{ and } (u_{ij}^*) \text{ are unitaries}, (R_G)\big),$$

where we omit the commutativity of the generators u_{ij}.

Using this idea, Wang [230, 231] constructed three *free quantum groups*, namely the free orthogonal O_n^+, the free unitary U_n^+, and the free symmetric quantum group S_n^+, liberating the groups O_n, U_n and S_n. A further example is the free hyperoctahedral quantum group H_n^+ introduced by Banica, Bichon, and Collins [13].

The intertwiner spaces of S_n, S_n^+, O_n, O_n^+, H_n and H_n^+ admit a combinatorial description by means of partitions. The process of liberation is reflected by restricting to those partitions that are non-crossing. In their 2009 article [24], Banica and Speicher initiated a systematic study of easy quantum groups, i.e.

of those compact quantum groups whose intertwiner spaces are described by the combinatorics of categories of partitions (see Definition 6.3 of [24] or Definition 1.4 of [232]). This class of quantum groups includes S_n^+, O_n^+, and H_n^+, as well as the groups S_n, O_n and H_n, but it goes far beyond the question of liberation of groups. Roughly speaking, it contains all compact quantum groups G with $S_n \subset G \subset O_n^+$, whose intertwiner spaces "have a nice combinatorics". It is a consequence of the seminal work by Woronowicz [234] that the correspondence between easy quantum groups and their categories of partitions is one-to-one.

The work on easy quantum groups has been continued by Banica, Bichon, Curran, Skalski, Sołtan, Speicher, Vergnioux, and the authors of the present chapter in a couple of articles [25, 26, 17, 18, 182, 16, 14, 23]. They have three aspects: firstly, easy quantum groups form a natural link between quantum groups, combinatorics and free probability theory [129, 57, 58, 19]. Secondly, they give rise to interesting operator algebras [218, 42, 91, 119]. Thirdly, the approach of Banica and Speicher via easy quantum groups systematised the study of free quantum groups in an accessible framework, which gives rise to a large number of new examples of compact quantum groups. Amongst others, it lead to the discovery of further examples of free quantum groups (see Theorem 3.16 of [24] and Corollary 2.10 of [232]). These free easy quantum groups (also called free orthogonal quantum groups) and likewise the easy groups were completely classified by Banica and Speicher [24], and by the second author [232]. Furthermore, examples of half-liberated easy quantum groups were given by Banica, Curran, Speicher, and the second author [24, 17, 232], and they were completely classified in [232]. The half-liberation is given by replacing the commutation relation

$$u_{ij}u_{kl} = u_{kl}u_{ij}$$

by

$$u_{ij}u_{kl}u_{rs} = u_{rs}u_{kl}u_{ij} ,$$

which can be interpreted as a slight weakening of commutativity.

It quickly turned out, that there are even more easy quantum groups than the above mentioned – and in this chapter, we show that there are in fact uncountably many and that they cannot be classified. While the classification of non-hyperoctahedral easy quantum groups is complete [17, 232], the case of hyperoctahedral easy quantum groups was still open. Hyperoctahedral easy quantum groups are quantum subgroups of H_n^+ corresponding to hyperoctahedral categories of partitions, i.e. categories which contain the four block partition ⊓⊓⊓ (four points, which are all connected) but not the double singleton $\uparrow \otimes \uparrow$ (two points, which are not connected). See Section 6.1.2 for details on partitions and categories.

We isolate a natural class of hyperoctahedral easy quantum groups – which we call *simplifiable* – with the commutation relations

$$u_{ij}^2 u_{kl} = u_{kl} u_{ij}^2 \,.$$

By this, the focus is put onto a quite unexplored class of partitions, and new questions arise. The main feature of these quantum groups is that the squares of the generators u_{ij} commute, whereas the elements u_{ij} itself behave rather like free elements. This mixture of commutative and non-commutative structures could play a special role in the understanding of non-commutative distributions. In this context, our result that the lattice of varieties of groups embeds into the lattice of simplifiable quantum groups can be interpreted as an indicator for the fact that the latter offer a rich source of new phenomena in free probability theory.

The technical heart of this chapter is worked out in Sections 6.2 and 6.3, where we construct a map from simplifiable categories of partitions to subgroups of $\mathbb{Z}_2^{*\infty}$. Given such a category $\mathcal{C}$, we label the partitions in $\mathcal{C}$ according to their block structure by letters $a_1, a_2, \ldots$ in order to obtain words. Mapping these words to $\mathbb{Z}_2^{*\infty}$ (where now $a_i^2 = e$), we obtain the following main result:

Theorem 6.A (See Theorem 6.3.10)**.** *There is a lattice isomorphism between simplifiable categories of partitions and proper S_0-invariant subgroups of E, where E is the subgroup of $\mathbb{Z}_2^{*\infty}$ consisting of all words of even length.*

Here, S_0 is the subsemigroup of $\mathrm{End}(\mathbb{Z}_2^{*\infty})$ generated by all inner automorphisms and by finite identifications of letters. This way, we obtain a one-to-one correspondence with a class of invariant subgroups of $\mathbb{F}_\infty$, which contains the lattice of fully characteristic subgroups of $\mathbb{F}_\infty$. This lattice in turn is anti-isomorphic to the lattice of varieties of groups [144]. See Section 6.1.4 for an introduction to varieties of groups and fully characteristic subgroups. By Olshanskii [148], there are uncountably many varieties of groups. Hence, we derive the following theorem.

Theorem 6.B (See Theorems 6.4.7, and 6.4.9)**.** *There is an injection of lattices of varieties of groups into the lattice of easy quantum groups. In particular, there are uncountably many easy quantum groups that are pairwise non-isomorphic.*

We express the relation between S_0-invariant proper subgroups of E and their associated simplifiable quantum groups by means of a quantum isometry group construction. A quantum isometry group is the maximal quantum group acting faithfully by isometries on a non-commutative space. It is the non-commutative replacement of the isometry group. Quantum isometry groups were studied by Bichon [34], Banica [10, 11], Goswami [102], Bhowmick and Goswami [33, 32], Banica and Skalski [20], Quaegebeur and Sabbe [180] and others. Banica and Skalski first studied the quantum isometry groups of discrete group duals in [22]. Other examples of such quantum isometry groups where studied by Liszka-Dalecki and Sołtan [132] and Tao and Qiu [203]. Notably, Banica and Skalski related in [21] quantum isometry groups and easy quantum groups for the first time.

If H is an S_0-invariant subgroup of $\mathbb{Z}_2^{*\infty}$, denote by $(H)_n$ the set of all words in H that involve at most the first n letters of $\mathbb{Z}_2^{\infty}$. Denote by $H_n^{[\infty]}$ the maximal simplifiable easy quantum group.

Theorem 6.C (See Theorems 6.6.3 and 6.6.6)**.** *If $H \leqslant E \leqslant \mathbb{Z}_2^{*\infty}$ is a proper S_0-invariant subgroup of E, then*

$$H_n^{[\infty]} \cap \mathrm{QISO}(C^*(\mathbb{Z}_2^{*n}/(H)_n))$$

is a simplifiable easy quantum group.

Vice versa, the diagonal subgroup of any simplifiable easy quantum group is of the form $\mathbb{Z}_2^{*n}/(H)_n$ for some proper S_0-invariant subgroup $H \leqslant E$. Moreover, these two operations are inverse to each other.

This correspondence in connection with Theorem 6.B, yields a large class of examples of non-classical quantum isometry groups.

6.1 Preliminaries and notations

In the whole chapter, tensor products of C*-algebras are taken with respect to the minimal C*-norm.

6.1.1 Compact quantum groups and compact matrix quantum groups

In [236], Woronowicz defines a *compact quantum group* (CQG) as a unital C*-algebra A with a unital *-homomorphism $\Delta : A \to A \otimes A$ such that

- Δ is *coassociative*, i.e. $(\Delta \otimes \mathrm{id}) \circ \Delta = (\mathrm{id} \otimes \Delta) \circ \Delta$,
- (A, Δ) is *bisimplifiable*, i.e. the subspaces $\operatorname{span} \Delta(A)(1 \otimes A)$ and $\operatorname{span} \Delta(A)(A \otimes 1)$ are dense in $A \otimes A$.

If (A, Δ) is a CQG, then Δ is called its *comultiplication*. Note that the bisimplifiability condition is in fact an assumption on left and right cancellation (see [236, Remark 3] or [205, Proof of Proposition 5.1.3]). All quantum groups in this chapter are universal versions, meaning that it is as a C*-algebra isomorphic to the universal enveloping C*-algebra of its polynomial subalgebra (see [205, Chapter 5.4]). A morphism between two CQGs in their universal version A and B is a unital *-homomorphism $\phi : A \to B$ such that $(\phi \otimes \phi) \circ \Delta_A = \Delta_B \circ \phi$. We say that A is a *quantum subgroup* of B if there is a surjective morphism $B \twoheadrightarrow A$, and they are *isomorphic* if there is a bijective morphism between them.

A *unitary corepresentation matrix* of A is a unitary element $u \in \mathrm{M}_n(A)$ such that $\Delta_A(u_{ij}) = \sum_k u_{ik} \otimes u_{kj}$ for all $1 \leqslant i, j \leqslant n$.

The concept of CQGs evolved from *compact matrix quantum group* (CMQG), [233, 235]. A compact matrix quantum group is a unital C*-algebra A with an element $u \in \mathrm{M}_n(A)$ such that

- A is generated by the entries of u,
- there is a *-homomorphism $\Delta : A \to A \otimes A$ such that $\Delta(u_{ij}) = \sum_k u_{ik} \otimes u_{kj}$ for all $1 \leqslant i, j \leqslant n$,
- u and its transpose u^t are invertible.

Every CMQG gives rise to a CQG, but the former contains more information – the choice of u. The matrix u is called the *fundamental corepresentation* of (A, u) and it is a corepresentation matrix of (A, Δ). A CMQG (A, u) is a universal version, if and only if A is the universal enveloping C*-algebra of the *-algebra generated by the entries of u. A morphism between CMQGs in their universal version A and B is a morphism of the underlying CQGs such that $(\phi \otimes \mathrm{id})(u_A)$ is conjugate by a matrix in $\mathrm{GL}_n(\mathbb{C})$ with u_B. If A and B are CMQGs and there is a bijective morphism of CMQGs between them, we say that they are *similar*. We say that two CMQGs are *isomorphic* if they are isomorphic as CQGs.

6.1.2 Tannaka-Krein duality, easy quantum groups and categories of partitions

Woronowicz' Tannaka-Krein duality

If (A, Δ) is a CQG and $u \in \mathrm{M}(\mathscr{K}(H) \otimes A)$ lies in the multiplier algebra of $\mathscr{K}(H) \otimes A$ for some Hilbert space H, then u is a *unitary corepresentation* of A if

u is a unitary and $(\mathrm{id} \otimes \Delta)(u) = u_{12}u_{13}$. We used the leg notation: $u_{12} = u \otimes 1$, $u_{13} = (\mathrm{id} \otimes \Sigma)(u \otimes 1)$, where Σ is the flip on $A \otimes A$. A morphism between unitary corepresentations $u \in \mathrm{M}(\mathscr{K}(H) \otimes A)$ and $v \in M(\mathscr{K}(K) \otimes A)$ is a bounded linear operator $T \in \mathscr{B}(H, K)$ such that $(T \otimes 1) \circ u = v \circ (T \otimes 1)$. A morphism between two unitary corepresentations is also called an *intertwiner*. The space of intertwiners between two unitary corepresentations $u \in A \otimes \mathscr{B}(H)$ and $v \in A \otimes \mathscr{B}(K)$ is denoted by $\mathrm{Hom}(u, v)$. With this structure, the finite dimensional unitary corepresentations of a CQG (A, Δ) form the *concrete C^*-category* $\mathrm{UCorep}_{\mathrm{fin}}(A)$, i.e. a C*-category with a faithful C*-functor $\mathrm{UCorep}_{\mathrm{fin}} \to \mathrm{FdHilb}$ to the category of finite dimensional Hilbert spaces (see [234] for details). The *tensor product* of two corepresentation $u \in \mathscr{B}(H) \otimes A$ and $v \in \mathscr{B}(K) \otimes A$ is defined by $u \otimes v = u_{13}v_{23}$. This tensor product induces the structure of a *concrete complete compact tensor C^*-category* in the sense of Woronowicz on $\mathrm{UCorep}_{\mathrm{fin}}(A)$ (see [233, 234] or [205, Chapter 5]).

The fundamental corepresentation of a CMQG is a generator of its category of finite dimensional corepresentations.

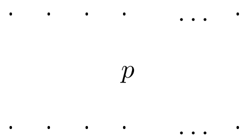
Theorem 6.1.1 (See Proposition 6.1.6 of [205]). *If v is a unitary corepresentation of a compact matrix quantum group (A, u), then there is $k \in \mathbb{N}$ such that v is a subobject of $u^{\otimes k}$.*

Woronowicz proved the following version of Tannaka-Krein duality.

Theorem 6.1.2 (See [234]). *Any concrete complete compact tensor C^*-category arises as the category of finite dimensional unitary corepresentations of some compact quantum group. Two compact quantum groups A and B are isomorphic if and only if their categories of corepresentations are equivalent over* FdHilb*. If A and B are compact matrix quantum groups, they are similar if and only if their categories of corepresentations are equivalent over* FdHilb *by a functor preserving the isomorphism class of the fundamental corepresentation.*

Categories of partitions

In order to describe corepresentation categories of quantum groups combinatorially, Banica and Speicher introduced the notions of a *category of partitions* and of *easy quantum groups* [24]. A *partition p* is given by k upper points and l lower points which may be connected by lines. By this, the set of $k + l$ points is partitioned into several *blocks*. We write a partition as a diagram in the following way:



k upper points and
l lower points.

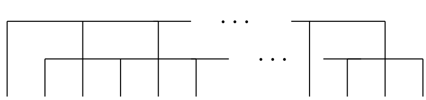
Two examples of such partitions are the following diagrams.

In the first example, all four points are connected, and the partition consists only of one block. In the second example, the left upper point and the right lower point are connected, whereas neither of the two remaining points is connected to any other point.

The set of partitions on k upper and l lower points is denoted by $P(k,l)$, and the set of all partitions is denoted by P. A partition $p \in P(k,l)$ is called *non-crossing*, if it can be drawn in such a way that none of its lines cross.

A few partitions play a special role, and they are listed here:

- The *singleton partition* $\uparrow$ is the partition in $P(0,1)$ on a single lower point.
- The *double singleton partition* $\uparrow \otimes \uparrow$ is the partition in $P(0,2)$ on two non-connected lower points.
- The *pair partition* (also called *duality partition*) ⊓ is the partition in $P(0,2)$ on two connected lower points.
- The *unit partition* (also called *identity partition*) $|$ is the partition in $P(1,1)$ connecting one upper with one lower point.
- The *four block partition* ⊓⊓⊓ is the partition in $P(0,4)$ connecting four lower points.
- The *s-mixing partition* h_s is the partition in $P(0,2s)$ for $s \in \mathbb{N}$ given by two blocks connecting the $2s$ points in an alternating way:

$h_s =$ 

- The *crossing partition* (also called *symmetry partition*) ╳ is the partition in $P(2,2)$ connecting the upper left with the lower right point, as well as the upper right point with the lower left one. It is the partition of two crossing pair partitions.
- The *half-liberating partition* ⋇ is the partition in $P(3,3)$ given by the blocks $\{1,3'\}$, $\{2,2'\}$ and $\{3,1'\}$ connecting three upper points $1,2,3$ and three lower points $1',2',3'$ such that 1 and $3'$ are connected, 2 and $2'$, and finally 3 and $1'$.

Further partitions will be introduced in Section 6.2.2.

We will also use *labelled partitions*, i.e. partitions whose points are either labelled by numbers or by letters. The labelling of a partition $p \in P(k,l)$ with letters is usually proceeded by starting at the very left of the k upper points of p and then going clockwise, ending at the very left of the l lower points. The labelling with numbers typically labels both the upper and the lower row of points from left to right.

There are the natural operations *tensor product* $(p \otimes q)$, *composition* (pq), *involution* (p^*) and *rotation* on partitions (see [24, Definition 1.8] or [232, Definition 1.4]). A collection $\mathcal{C}$ of subsets $D(k,l) \subset P(k,l)$, $k,l \in \mathbb{N}$ is called a *category of partitions* if it is closed under these operations and if it contains the pair partition ⊓, and the unit partition | (see [24, Definition 6.1] or [232, Definition 1.4]).

A category of partitions $\mathcal{C}$ is called *hyperoctahedral* if the four block ⊓⊓⊓ is in $\mathcal{C}$, but the double singleton $\uparrow \otimes \uparrow$ is not in $\mathcal{C}$.

Given a partition $p \in P(k,l)$ and two multi-indices $(i_1,\dots,i_k)$, $(j_1,\dots,j_l)$, we can label the diagram of p with these numbers (now, the upper and the lower row both are labelled from left to right, respectively) and we put

$$\delta_p(i,j) = \begin{cases} 1 & \text{if } p \text{ connects only equal indices,} \\ 0 & \text{if there is a string of } p \text{ connecting unequal indices.} \end{cases}$$

For every $n \in \mathbb{N}$, there is a map $T_p : (\mathbb{C}^n)^{\otimes k} \to (\mathbb{C}^n)^{\otimes l}$ associated with p, which is given by

$$T_p(e_{i_1} \otimes \cdots \otimes e_{i_k}) = \sum_{1 \leqslant j_1,\dots,j_l \leqslant n} \delta_p(i,j) \cdot e_{j_1} \otimes \cdots \otimes e_{j_l} .$$

Definition 6.1.3 (Definition 6.1 of [24] or Definition 2.1 of [17])**.** A compact matrix quantum group (A,u) is called *easy*, if there is a category of partitions $\mathcal{C}$ given by $D(k,l) \subset P(k,l)$, for all $k,l \in \mathbb{N}$ such that

$$\mathrm{Hom}(u^{\otimes k}, u^{\otimes l}) = \mathrm{span}\{T_p \,|\, p \in D(k,l)\} .$$

Combining Theorems 6.1.1 and 6.1.2, we obtain the following theorem, which is the basis of all combinatorial investigation on easy quantum groups.

Theorem 6.1.4 (See [24]). *There is a bijection between categories of partitions and easy quantum groups up to similarity.*

Thus, easy quantum groups are completely determined by their categories of partitions.

6.1.3 Quantum isometry groups

Given a discrete group G with finite generating set $S \subset G$ and associated word-length function $l : G \to \mathbb{N}$, $l(g) = \min\{n \in \mathbb{N} \mid \exists s_1, \dots, s_n \in S : g = s_1 \cdots s_n\}$, we obtain a quantum isometry group of $\mathrm{C}^*_{\max}(G)$ along the lines of [20]. We denote by u_g the canonical unitary of $\mathrm{C}^*_{\max}(G)$ associated with $g \in G$.

Definition 6.1.5 (Definitions 2.5 and Section 4 of [20]). Let $(A, u = (u_{st})_{s,t \in S})$ be a compact matrix quantum group and write $\mathrm{Pol}(A)$ for its polynomial algebra $* - \mathrm{alg}(u_{st} \mid s, t \in S)$. An action $\alpha : \mathrm{C}^*_{\max}(G) \to \mathrm{C}^*_{\max}(G) \otimes A$ on $\mathrm{C}^*_{\max}(G)$ is faithful and isometric with respect to l, if

- $\alpha(L_n) \subset L_n \otimes \mathrm{Pol}(A)$ for all $n \in \mathbb{N}$, where $L_n = \mathrm{span}\{u_g \mid l(g) = n\}$ and
- $\alpha(u_s) = \sum_{t \in S} u_t \otimes u_{ts}$ for all $s \in S$.

Theorem 6.1.6 (Theorems 2.7 and 4.5 of [20]). *There is a maximal compact matrix quantum group (A, u) acting faithfully and isometrically with respect to l on $C^*_{\max}(G)$. That is, for any other compact matrix quantum group (B, v) acting faithfully and isometrically with respect to l on $C^*_{\max}(G)$ there is a unique morphism of CMQGs $\phi : (A, u) \to (B, v)$ such that $\phi(u) = v$.*

6.1.4 Varieties of groups

In this section we briefly explain the concepts of varieties of groups. We advice the interested reader to consult [144] for a thorough introduction.

Consider $\mathbb{F}_\infty$ with free basis $x_1, x_2, \dots$ and let $w \in \mathbb{F}_\infty$ be a word in the letters $x_1, x_2, \dots x_n$. We say that the *identical relation* w holds in a group G if for any choice of elements $g_1, g_2, \dots g_n \in G$, replacing x_i by g_i, we have $w(g_1, \dots, g_n) = 1_G$. Following [144] a *variety of groups* $\mathcal{V}$ is a class of groups for which there is a set of words $R \subset \mathbb{F}_\infty$ such that every group G in $\mathcal{V}$ satisfies the identical relations in R.

Let us give some examples of varieties of groups.

Example 6.1.7. The following classes of groups are varieties of groups. We also describe the identical relations that characterise them.

1. The class of all groups is the variety of groups, where no law is satisfied.
2. The class of abelian groups is defined by the commutator $[x, y] = xyx^{-1}y^{-1}$.
3. The class of groups with a fixed exponent s is given by x^s.
4. The class of nilpotent groups of class 2 is described by $[[x, y], z]$.

Varieties of groups are important for this work, because they correspond precisely to the *fully characteristic subgroups* of $\mathbb{F}_\infty$. Given an inclusion of groups $H \leqslant G$, H is fully characteristic in G, if it is invariant under all endomorphisms of G. This means that $\phi(H) \subset H$ for every endomorphism $\phi \in \mathrm{End}(G)$.

The set of identical relations that hold in a given group, form a subgroup of $\mathbb{F}_\infty$. This observation is the trigger to prove the following theorem.

Theorem 6.1.8 (See [143] or Theorem 14.31 in [144])**.** *There is a lattice anti-isomorphism between varieties of groups and fully characteristic subgroups of $\mathbb{F}_\infty$ sending a variety of groups to the set of all identical relations that hold in it.*

We will make use of another observation concerning elements of free groups. Two sets of words in $\mathbb{F}_n$ are called *equivalent*, if they generate the same fully characteristic subgroup.

Theorem 6.1.9 (See Theorem 12.12 in [144])**.** *Every word $w \in \mathbb{F}_n$, $n \in \mathbb{N} \cup \{\infty\}$ is equivalent to a pair of words a and b in $\mathbb{F}_n$, where a is of the form x^m for some $m \geqslant 2$ and $x \in \mathbb{F}_n$, and b is an element of the commutator subgroup $[\mathbb{F}_n, \mathbb{F}_n]$.*

6.2 Simplifiable hyperoctahedral categories

6.2.1 A short review of the classification of easy quantum groups

Recall from Section 6.1.2 that a category of partitions is called *hyperoctahedral*, if it contains the four block partition ⊓⊓⊓ but not the partition $\uparrow \otimes \uparrow$. An easy quantum group G is called hyperoctahedral, if its corresponding category of partitions is hyperoctahedral. By [17, Theorem 6.5] and [232, Corollary 4.11] we know that there are exactly 13 non-hyperoctahedral easy quantum groups, resp. 13 non-hyperoctahedral categories of partitions, so they are completely classified. We will shed some light on the classification of hyperoctahedral categories. Let us first give a short review of the classification of easy quantum groups.

For partitions $p_1, \dots, p_n \in P$, we write $\mathcal{C} = \langle p_1, \dots, p_n \rangle$ for the category generated by these partitions, i.e. $\mathcal{C}$ is the smallest subclass of P which is closed under the category operations (see Section 6.1.2) and which contains the partitions $p_1, \dots, p_n$. (Note that the pair partition $\sqcap$ and the unit partition $|$ are always contained in a category as trivial base cases.)

By [24, Theorem 3.16] and [232, Corollary 2.10], there are exactly seven free easy quantum groups (also called free orthogonal quantum groups), namely:

$$\begin{array}{ccccccc} B_n^+ & \subset & {B'_n}^+ & \subset & B_n^{\#+} & \subset & O_n^+ \\ \cup & & \cup & & & & \cup \\ S_n^+ & \subset & {S'_n}^+ & & \subset & & H_n^+ . \end{array}$$

The corresponding seven categories of partitions are described as follows.

$$\begin{array}{ccccccc} \langle \uparrow \rangle & \supset & \langle \shortmid \sqcap\!\sqcap \rangle & \supset & \langle \uparrow \otimes \uparrow \rangle & \supset & \langle \varnothing \rangle = NC_2 \\ \cap & & \cap & & & & \cap \\ \langle \uparrow, \sqcap\!\sqcap\!\sqcap \rangle = NC & \supset & \langle \uparrow \otimes \uparrow, \sqcap\!\sqcap\!\sqcap \rangle & & \supset & & \langle \sqcap\!\sqcap\!\sqcap \rangle . \end{array}$$

Note that these partitions are all non-crossing, i.e. all of these seven categories are subclasses of NC, the collection of all non-crossing partitions. We denote by NC_2 the category of all non-crossing pair partitions. Furthermore, note that only $\langle \sqcap\!\sqcap\!\sqcap \rangle$ is a hyperoctahedral category, the category corresponding to the hyperoctahedral quantum group H_n^+ by [13]. The other six categories are non-hyperoctahedral.

Besides the non-crossing categories, there are many categories which contain partitions that have some crossing lines. The most prominent partition which involves a crossing is the *crossing partition* (also called *symmetry partition*) $\times$

in $P(2,2)$. Every category containing the crossing partition corresponds to a group. By [24, Theorem 2.8] we know that there are exactly six easy groups.

$$\begin{array}{ccccc} B_n & \subset & B_n' & \subset & O_n \\ \cup & & \cup & & \cup \\ S_n & \subset & S_n' & \subset & H_n. \end{array}$$

Accordingly, there are exactly six categories of partitions containing the crossing partition $\times$.

$$\begin{array}{ccccc} \langle \times, \uparrow \rangle & \supset & \langle \times, \uparrow \otimes \uparrow \rangle & \supset & \langle \times \rangle = P_2 \\ \cap & & \cap & & \cap \\ \langle \times, \uparrow, \sqcap\!\sqcap\!\sqcap \rangle = P & \supset & \langle \times, \uparrow \otimes \uparrow, \sqcap\!\sqcap\!\sqcap \rangle & \supset & \langle \times, \sqcap\!\sqcap\!\sqcap \rangle. \end{array}$$

Note that on the level of categories containing the crossing partition, the two categories $\langle \times, \shortmid \sqcap\!\sqcap \rangle$ and $\langle \times, \uparrow \otimes \uparrow \rangle$ coincide. Furthermore, amongst the above categories only $\langle \times, \sqcap\!\sqcap\!\sqcap \rangle$ is hyperoctahedral; the other five categories are non-hyperoctahedral.

Half-liberated easy quantum groups were introduced in [24] and [17]. They correspond to categories containing the half-liberating partition $\ast$ but not the crossing partition $\times$. By [232, Theorem 4.13], there are exactly the following half-liberated easy quantum groups, containing the *hyperoctahedral series* $H_n^{(s)}$, $s \geqslant 3$ of [17, Definition 3.1].

$$\begin{array}{ccc} B_n^{\#*} & \subset & O_n^* \\ & & \cup \\ & & H_n^* \\ & & \cup \\ & & H_n^{(s)}, s \geqslant 3. \end{array}$$

The corresponding categories of partitions are described as follows.

$$\begin{array}{ccc}
\langle \ast, \uparrow \otimes \uparrow \rangle & \supset & \langle \ast \rangle \\
 & & \cap \\
 & & \langle \ast, \sqcap\sqcap\sqcap \rangle \\
 & & \cap \\
 & & \langle \ast, \sqcap\sqcap\sqcap, h_s \rangle.
\end{array}$$

Here, $\langle \ast, \sqcap\sqcap\sqcap \rangle$ and $\langle \ast, \sqcap\sqcap\sqcap, h_s \rangle$ are hyperoctahedral, for all $s \geqslant 3$. The categories $\langle \ast, \uparrow \otimes \uparrow \rangle$ and $\langle \ast \rangle$ in turn are two more non-hyperoctahedral categories, completing the list of 13 non-hyperoctahedral categories.

We conclude that the only class of categories which ought to be classified is the one of hyperoctahedral categories, as illustrated by the following picture.

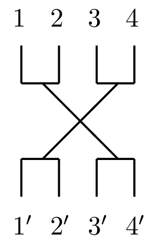
$$\begin{array}{ccccc}
\langle \uparrow, \sqcap\sqcap\sqcap \rangle & \supset & \langle \uparrow \otimes \uparrow, \sqcap\sqcap\sqcap \rangle & \supset & \langle \sqcap\sqcap\sqcap \rangle \\
 & & & & \cap \\
\cap & & & & ? \\
 & & & \not\supset & \cap \\
\langle \diagup\!\!\!\!\diagdown, \uparrow, \sqcap\sqcap\sqcap \rangle = P & \supset & \langle \diagup\!\!\!\!\diagdown, \uparrow \otimes \uparrow, \sqcap\sqcap\sqcap \rangle & \supset & \langle \diagup\!\!\!\!\diagdown, \sqcap\sqcap\sqcap \rangle.
\end{array}$$

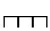
The question is to find all categories $\mathcal{C}$ of partitions, which contain the four block $\sqcap\sqcap\sqcap$ but not the double singleton $\uparrow \otimes \uparrow$. Furthermore, we can restrict to those categories which do not contain the half-liberating partition $\ast$. The *higher hyperoctahedral series* $H_n^{[s]}$, $s \in \{3, 4, \ldots, \infty\}$ of [17, Section 4] fall into this class. They are given by the categories $\langle \sqcap\sqcap\sqcap, h_s \rangle$.

6.2.2 Base cases in the class of hyperoctahedral categories

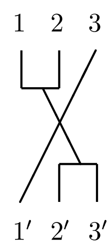
By definition, the category $\langle \sqcap\sqcap\sqcap \rangle$ is a natural base case in the class of hyperoctahedral categories, but we will see that also other categories serve as base cases for interesting subclasses of hyperoctahedral categories. For this, we introduce two more partitions.

Definition 6.2.1. The *fat crossing partition*  is the following partition in $P(4,4)$, connecting the upper points 1 and 2 with the lower points $3'$ and $4'$, as well as the upper points 3 and 4 with the lower points $1'$ and $2'$, i.e.  consists of two crossing four blocks.



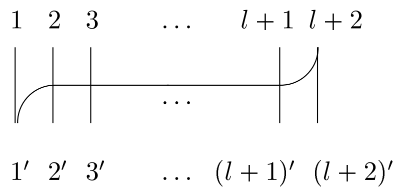
Note that any category $\mathcal{C}$ containing the fat crossing, also contains the four block partition (see also Lemma 6.2.4). If furthermore $\uparrow \otimes \uparrow \notin \mathcal{C}$, then $\mathcal{C}$ is hyperoctahedral. The converse is also true: Any hyperoctahedral category (apart from $\langle$$\rangle$) contains the fat crossing (see Proposition 6.2.12).

Definition 6.2.2. The *pair positioner partition*  is the following partition in $P(3,3)$, consisting of a four block on 1, 2, $2'$ and $3'$ and a pair on 3 and $1'$.



Remark 6.2.3. In [17, Lemma 4.2] the following partitions $k_l \in P(l+2, l+2)$ for $l \in \mathbb{N}$ were used to define the *higher hyperoctahedral series* $H_n^{[s]}$ (see also [232] for a definition of k_l). They are given by a four block on $\{1, 1', l+2, (l+2)'\}$ and pairs on $\{i, i'\}$ for $i = 2, \ldots, l+1$. The following picture illustrates the partition k_l – note that the waved line from $1'$ to $l+2$ is *not* connected to the lines from 2 to $2'$, from 3 to $3'$ etc.

$k_l =$



We check that k_1 is in a category $\mathcal{C}$ if and only if all k_l are in $\mathcal{C}$ for all $l \in \mathbb{N}$ (apply the pair partition to $k_l \otimes k_1$ to obtain k_{l+1}). The pair positioner partition ⊔⁄⊓⊓ is a rotated version of k_1, thus ⊔⁄⊓⊓ $\in \mathcal{C}$ if and only if $k_1 \in \mathcal{C}$. Furthermore, ⟨⊔⁄⊓⊓⟩ corresponds to $H_n^{[\infty]}$ of [17].

The fat crossing partition $\overset{\sqcup\sqcup}{\underset{\sqcap\sqcap}{\times}}$ can be constructed out of the pair positioner partition ⊔⁄⊓⊓ using the category operations. The following lemma shows some relations between the partitions.

Lemma 6.2.4. *The following partitions may be generated inside the following categories using the category operations.*

(i) ⊓⊓⊓ $\in \langle \overset{\sqcup\sqcup}{\underset{\sqcap\sqcap}{\times}} \rangle$.

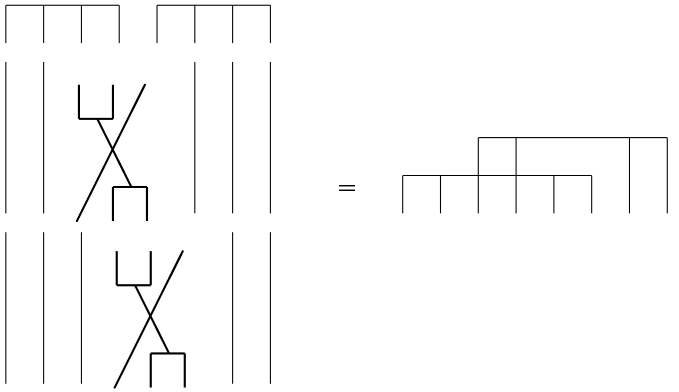
(ii) $\overset{\sqcup\sqcup}{\underset{\sqcap\sqcap}{\times}} \in$ ⟨⊔⁄⊓⊓⟩.

(iii) ⊔⁄⊓⊓ $\in \langle h_s \rangle$ *for all* $s \geqslant 3$.

(iv) ⊔⁄⊓⊓ $\in$ ⟨⁎, ⊓⊓⊓⟩.

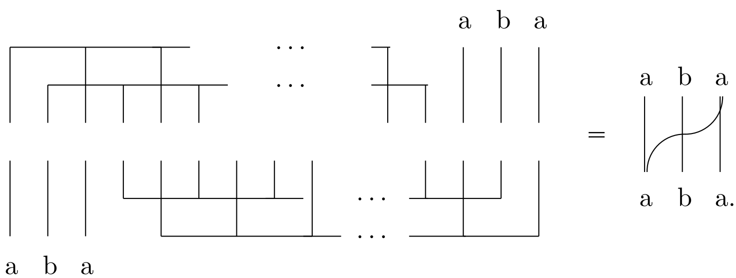
Proof. (i) We obtain ⊓⊓⊓ as the composition of $\overset{\sqcup\sqcup}{\underset{\sqcap\sqcap}{\times}}$, $| \otimes \sqcap \otimes |$ and $\sqcap$.

(ii) Compose the tensor product ⊓⊓⊓ $\otimes$ ⊓⊓⊓ with ⊔⁄⊓⊓ in the following way:



Then use rotation to obtain $\sqcup\sqcup/\sqcap\sqcap$ (crossed).

(iii) We construct the rotated version k_1 of $\sqcup\!/\!\sqcap\sqcap$ using $h_s \otimes |^{\otimes 3}$ and its rotated version:



(iv) The partition $\overset{\sqcup}{\sqcap}\in P(2,2)$ is a rotated version of the four block $\sqcap\!\sqcap$. Compose $\overset{\sqcup}{\sqcap}\otimes|$ with the half-liberated partition $\ast$ to obtain $\sqcup\!/\!\sqcap\sqcap$. $\square$

The pair positioner partition $\sqcup\!/\!\sqcap\sqcap$ plays an important role in the sequel. By the preceding lemma, we see that any category $\mathcal{C}$ containing the pair positioner partition $\sqcup\!/\!\sqcap\sqcap$ also contains the four block partition $\sqcap\!\sqcap$. Thus, these categories form a subclass of the hyperoctahedral categories.

Definition 6.2.5. A category of partitions is called *simplifiable* if it contains the pair positioner partition $\sqcup\!/\!\sqcap\sqcap$ but not the double singleton $\uparrow\otimes\uparrow$.

Note that every simplifiable category is also hyperoctahedral. Simplifiable categories carry a nice feature – they can be described by very simplified partitions. This is the content of Lemma 6.2.7. We first prove a lemma on the block structure of partitions in simplifiable categories.

Lemma 6.2.6. *Let $\mathcal{C}$ be any category of partitions, and let $p \in \mathcal{C}$.*

(a) If $\mathcal{C}$ contains the four block partition $\sqcap\!\sqcap\!\sqcap$, we can connect neighbouring blocks of p inside of $\mathcal{C}$, i.e. the partition p' obtained from p by connecting two blocks of p which have at least two neighbouring points is again in $\mathcal{C}$.

(b) If $\mathcal{C}$ contains the pair positioner partition $\sqcup\!\!/\!\sqcap\!\sqcap$, we can connect arbitrary blocks of p inside of $\mathcal{C}$, i.e. the partition p' obtained from p by combining two arbitrary blocks of p is again in $\mathcal{C}$.

Proof. We may assume that p has no upper points, by rotation.

(a) We can compose p with $|^{\otimes\alpha} \otimes \genfrac{}{}{0pt}{}{\sqcup}{\sqcap} \otimes |^{\otimes\beta}$ for suitable α and β.

(b) By composition, we insert a pair partition $\sqcap$ next to the block b_1 of p. By (a), we can connect it to b_1. Using the pair positioner partition $\sqcup\!\!/\!\sqcap\!\sqcap$, we can shift these two points next to the block b_2. Again by (a), we connect it to b_2. This yields a partition in which the blocks b_1 and b_2 are connected. Capping this partition with the pair partition erases the two auxiliary points and yields the desired partition in $\mathcal{C}$. □

Let $p \in P(0,l)$ be a partition with k blocks. We may view p as a *word* in k letters $a_1, \ldots, a_k$ corresponding to the points connected by the partition p:

$$p = a_{i(1)}^{k_1} a_{i(2)}^{k_2} \ldots a_{i(n)}^{k_n}.$$

Here $a_{i(j)} \neq a_{i(j+1)}$ for $j = 1, \ldots, n-1$ and $k_j \in \mathbb{N}$. For example, the four block partition $\sqcap\!\sqcap\!\sqcap$ corresponds to the word a^4, (a rotated version of) the pair positioner partition $\sqcup\!\!/\!\sqcap\!\sqcap$ corresponds to ab^2ab^2, and the double singleton $\uparrow \otimes \uparrow$ corresponds to ab. Conversely, every word $a_{i(1)}^{k_1} a_{i(2)}^{k_2} \ldots a_{i(n)}^{k_n}$ of length l yields a partition $p \in P(0,l)$ connecting nothing but equal letters of the word.

For technical reasons, we introduce the *empty partition* $\emptyset \in P(0,0)$ which is by definition in any category of partitions $\mathcal{C}$.

Lemma 6.2.7. *Let $\mathcal{C}$ be a category of partitions. Let $p \in P(0,l)$ be a partition, seen as the word $p = a_{i(1)}^{k_1} a_{i(2)}^{k_2} \ldots a_{i(n)}^{k_n}$.*

(a) We put $k'_j := \begin{cases} 1 & \text{if } k_j \text{ is odd} \\ 2 & \text{if } k_j \text{ is even} \end{cases}$, *and* $p' := a_{i(1)}^{k'_1} a_{i(2)}^{k'_2} \ldots a_{i(n)}^{k'_n}$.

If $\mathcal{C}$ contains the four block partition ⊓⊓⊓, then $p \in \mathcal{C}$ if and only if $p' \in \mathcal{C}$.

(b) We put $k''_j := \begin{cases} 1 & \text{if } k_j \text{ is odd} \\ 0 & \text{if } k_j \text{ is even} \end{cases}$*, and* $p'' := a_{i(1)}^{k''_1} a_{i(2)}^{k''_2} \dots a_{i(n)}^{k''_n}$*. It is possible that* $p'' = \emptyset$*.*

If $\mathcal{C}$ contains the pair positioner partition ⊔/⊓, then $p \in \mathcal{C}$ if and only if $p'' \in \mathcal{C}$.

Proof. (a) If $k_j \geqslant 3$, we compose p with the pair partition to erase two of the neighbouring $a_{i(j)}$-points. Since this operation can be done iteratively and inside the category $\mathcal{C}$, we infer that $p' \in \mathcal{C}$ whenever $p \in \mathcal{C}$. For the converse, we compose p' with $|^{\otimes\alpha} \otimes \sqcap \otimes |^{\otimes\beta}$ for suitable α, β, such that the pair is situated right beside one of the $a_{i(j)}$-points of p'. By Lemma 6.2.6(a), we can connect these two points to the block to which $a_{i(j)}$ belongs, which yields a partition $\tilde{p}'$ where the power k_j of $a_{i(j)}$ is increased by two. By this procedure, we construct p out of p' inside the category $\mathcal{C}$.

(b) Assume first that $p'' \neq \emptyset$. If $p \in \mathcal{C}$, then $p'' \in \mathcal{C}$ again by using the pair partition $\sqcap$. For the converse, insert pair partitions $\sqcap$ at every position in p'' where $k''_j = 0$. By Lemma 6.2.6(b), we can connect these pairs to the according blocks of p'' such that we obtain a partition p' as in (a). Since the four block ⊓⊓⊓ is in $\mathcal{C}$ (see Lemma 6.2.4), we conclude $p \in \mathcal{C}$ using (a).

Secondly, if $p'' = \emptyset$, then all exponents k_j of p are even. All interval partitions $q = q_1 \otimes \dots \otimes q_m$, where every partition q_j consists of a single block of even length respectively, are in $\mathcal{C}$. Using the fat crossing partition ⊔⊔/⊓⊓ (which is in $\mathcal{C}$ by Lemma 6.2.4), the partition p may be obtained from a suitable interval partition q, by composition. Thus, $p \in \mathcal{C}$. □

Lemma 6.2.7(b) will be crucial for the study of simplifiable quantum groups in the sequel.

Remark 6.2.8. Lemma 6.2.7 can be extended to arbitrary partitions $p \in P(k, l)$, by rotation.

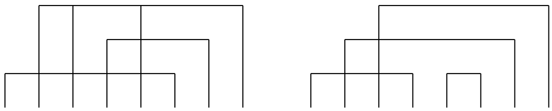
In simplifiable categories, we have a notion of equivalence of partitions according to Lemma 6.2.7(b).

Definition 6.2.9. Two partitions p, q are called *equivalent*, if q can be obtained from p by the following operations:

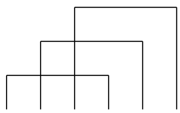
- Elimination of two consecutive points belonging to the same block.

- Insertion of two consecutive points into the partition at any position and either connecting it to any other block – or not.

Example 6.2.10. Both of the following partitions



are equivalent to the rotation of the half-liberating partition:



Lemma 6.2.11. *Let $\mathcal{C}$ be a simplifiable category and $p \in \mathcal{C}$. Let q be a partition that is equivalent to p. Then $q \in \mathcal{C}$.*

Proof. See Lemma 6.2.7(b). □

The category $\langle ⊔\!/\!⊓⊓ \rangle$ is the base case for the simplifiable categories of partitions, i.e. it is contained in all simplifiable categories. We show now that the category $\langle \overset{⊔⊔}{\underset{⊓⊓}{\times}} \rangle$ is a base case for all hyperoctahedral categories that contain at least one crossing partition.

Proposition 6.2.12. *Let $\mathcal{C}$ be a hyperoctahedral category of partitions with $\mathcal{C} \neq \langle ⊓⊤⊓ \rangle$. Then the fat crossing partition $\overset{⊔⊔}{\underset{⊓⊓}{\times}}$ is in $\mathcal{C}$.*

Proof. We show that one of the following cases hold for $\mathcal{C}$:

- $\overset{⊔⊔}{\underset{⊓⊓}{\times}} \in \mathcal{C}$.
- $⊔\!/\!⊓⊓ \in \mathcal{C}$.
- $h_s \in \mathcal{C}$ for some $s \geqslant 3$.

By Lemma 6.2.4 this will complete the proof.

The only hyperoctahedral category of *non-crossing* partitions is $\langle ⊓⊤⊓ \rangle$. Thus, $\mathcal{C}$ contains a partition $p \in P \backslash NC$ with a crossing. We may assume that p consists only of two blocks b_1 and b_2, after connecting all other blocks with one of the crossing blocks, using Lemma 6.2.6(a). Furthermore, we may assume that no three points in a row are connected by one of the blocks, by Lemma 6.2.7(a). Hence, we may write p as

$$p = a^{k_1} b^{k_2} a^{k_3} b^{k_4} \dots a^{k_n} \qquad \text{or} \qquad p = a^{k_1} b^{k_2} a^{k_3} b^{k_4} \dots b^{k_n}$$

where $k_i \in \{1,2\}$ and $n \geqslant 4$, and a and b correspond to the points connected by the blocks b_1 resp. b_2. Note that the length of p is even (otherwise we could construct the singleton $\uparrow$ using the pair partition).

If all $k_i = 1$, then $p = h_s$ for some $s \geqslant 2$ – the case $p = h_2$ implying all other cases of the claim. Otherwise, we may assume $k_1 = 2$ by rotation. If $n \geqslant 5$, we may erase the two points a^{k_1} using the pair partition and we obtain a partition $p' \in \mathcal{C}$ which still has a crossing. Iterating this procedure, we either end up with a partition h_s for some $s \geqslant 2$ or with a partition $p \in \mathcal{C}$ such that $k_1 = 2$ and $n = 4$. In the latter case, p is of length six or eight. There are exactly four cases of such a partition:

- $p = aababb$ – An application of the pair partition would yield $\uparrow \otimes \uparrow \in \mathcal{C}$ which is a contradiction.
- $p = aabaab$ – This is a rotated version of $\sqcup\!\!/\!\!\sqcap$.
- $p = aabbab$ – Again this would yield $\uparrow \otimes \uparrow \in \mathcal{C}$.
- $p = aabbaabb$ – This is $\overset{\sqcup\sqcup}{\underset{\sqcap\sqcap}{\times}}$ in a rotated version.

□

Remark 6.2.13. Since $\langle \overset{\sqcup\sqcup}{\underset{\sqcap\sqcap}{\times}} \rangle$ contains a crossing partition, we have $\langle \sqcap\!\sqcap\!\sqcap \rangle \subsetneq \langle \overset{\sqcup\sqcup}{\underset{\sqcap\sqcap}{\times}} \rangle$. Furthermore, we have $\langle \overset{\sqcup\sqcup}{\underset{\sqcap\sqcap}{\times}} \rangle \subset \langle \sqcup\!\!/\!\!\sqcap \rangle$ by Lemma 6.2.4. For the proof of $\langle \overset{\sqcup\sqcup}{\underset{\sqcap\sqcap}{\times}} \rangle \neq \langle \sqcup\!\!/\!\!\sqcap \rangle$ we refer to Section 6.5.

6.2.3 The single leg form of a partition

The pair positioner partition $\sqcup\!\!/\!\!\sqcap$ allows us to simplify the classification problem, since we can reduce to partitions of a nicer form.

Definition 6.2.14. A partition $p \in P$ is in *single leg form*, if p is – as a word – of the form

$$p = a_{i(1)} a_{i(2)} \dots a_{i(n)}\,,$$

where $a_{i(j)} \neq a_{i(j+1)}$ for $j = 1, \dots, n-1$. The letters $a_1, \dots, a_k$ correspond to the points connected by the partition p. In other words, in a partition in single leg form no two consecutive points belong to the same block.

Let $\mathcal{C}$ be a category of partitions (or simply a set of partitions). We denote by $\mathcal{C}_{sl}$ the set of all partitions $p \in \mathcal{C}$ in single leg form. By P_{sl}, we denote the collection of all partitions in single leg form.

Lemma 6.2.15. *Let $\mathcal{C}$ be a hyperoctahedral category and let $p \in \mathcal{C}_{sl}$ be a partition in single leg form. Then, every letter in the word p appears at least twice. Furthermore, every word in $\mathcal{C}_{sl}$ consists of at least two letters, it has length at least four, and it is of even length.*

Proof. The double singleton $\uparrow \otimes \uparrow$ is not contained in $\mathcal{C}$. □

If $p \in P(0,l)$ is a partition seen as the word $p = a_{i(1)}^{k_1} a_{i(2)}^{k_2} \dots a_{i(n)}^{k_n}$, the partition $p'' = a_{i(1)}^{k_1''} a_{i(2)}^{k_2''} \dots a_{i(n)}^{k_n''}$ of Lemma 6.2.7(b) is not necessarily in single leg form, e.g. $p = ab^2acacaca$ yields $p'' = a^2cacaca$. However, a finite iteration of the procedure as in Lemma 6.2.7(b) either yields a partition q in single leg form or the empty partition $\emptyset \in P(0,0)$. This partition q (possibly the empty partition) is called the *simplified partition associated to p*. Note that every partition has a unique simplified partition – the converse is not true. We can state a variation of Lemma 6.2.7(b).

Lemma 6.2.16. *Let $\mathcal{C}$ be a simplifiable category of partitions. Then, a partition is in $\mathcal{C}$ if and only if its simplified partition is in $\mathcal{C}$.*

Remark 6.2.17. Every partition $p \in P$ is equivalent to its simplified partition $p'' \in P$ in single leg form and two partitions are equivalent if and only if their simplified partitions agree. If $p \in P$ is in single leg form, then the simplified partition associated to p is p itself.

The set $\mathcal{C}_{sl}$ turns out to be a complete invariant for the simplifiable categories.

Proposition 6.2.18. *Let $\mathcal{C}$ and $\mathcal{D}$ be simplifiable categories.*

(a) The category $\langle \mathcal{C}_{sl}, \sqcup\!\!\!/\!\!\sqcap \rangle$ coincides with $\mathcal{C}$.

(b) We have $\mathcal{C}_{sl} = \mathcal{D}_{sl}$ if and only if $\mathcal{C} = \mathcal{D}$.

(c) We have $\mathcal{C}_{sl} \subseteq \mathcal{D}_{sl}$ if and only if $\mathcal{C} \subseteq \mathcal{D}$.

Proof. (a) We have $\langle \mathcal{C}_{sl}, \sqcup\!\!\!/\!\!\sqcap \rangle \subset \mathcal{C}$. On the other hand, if $p \in \mathcal{C}$, we consider its associated simplified partition $p'' \in \mathcal{C}$ by Lemma 6.2.16. Thus, $p'' \in \langle \mathcal{C}_{sl}, \sqcup\!\!\!/\!\!\sqcap \rangle$. Again by Lemma 6.2.16, we also have $p \in \langle \mathcal{C}_{sl}, \sqcup\!\!\!/\!\!\sqcap \rangle$. □

As a consequence, we can choose the generators of a simplifiable category always to be in single leg form. In the sequel, we will classify the subclass of simplifiable easy quantum groups by classifying the sets $\mathcal{C}_{sl}$.

6.3 A group theoretic framework for hyperoctahedral categories of partitions

Denote by $L = \{a_1, a_2, \dots\}$ an infinite countable number of letters. Let p be a partition with n blocks and choose a labelling $l = (a_{i(1)}, a_{i(2)}, \dots, a_{i(n)})$ of the blocks of p with pairwise different letters. Denote by $w(p,l)$ the word of $\mathbb{F}_L$ obtained by considering p as a word with letters given by l starting in the top left corner of p and going around clockwise. Note that mutually different blocks are labelled by mutually different letters. We write $G = \mathbb{Z}_2^{*L}$ for the infinite free product of the cyclic group of order 2 indexed by the letters in L. The canonical surjection $\mathbb{F}_L \twoheadrightarrow G$ is denoted by π.

The next observation describes the basic link between partitions and elements of G.

Lemma 6.3.1. *(i) Two partitions p and q are equivalent, if and only if $\pi(w(p,l)) = \pi(w(q,l'))$ for some labellings l and l'.*

(ii) Let $\mathcal{C}$ be a simplifiable category of partitions, and let $p \in \mathcal{C}$ with $\pi(w(p,l)) \neq e$ (where e denotes the neutral element in G). Then there is a partition $\emptyset \neq q \in \mathcal{C}$ in single leg form and a labelling l' such that $\pi(w(q,l')) = \pi(w(p,l))$.

Proof. (i) This follows from the fact, that two words w and v in $\mathbb{F}_L$ have the same image under π if and only if there is a sequence $w_1, \dots, w_n$ with $w_1 = w$ and $w_n = v$ such that w_{i+1} arises from w_i by inserting or deleting a square of a letter in L.

(ii) The simplified partition p'' associated to p is in $\mathcal{C}$ by Lemma 6.2.16. By (i) and Remark 6.2.17 we get the result. Note that $p'' \neq \emptyset$ since $\pi(w(p,l)) \neq e$. $\square$

Definition 6.3.2. Let $\mathcal{C}$ be a simplifiable category of partitions. We denote by $F(\mathcal{C})$ the subset of G formed by all elements $\pi(w(p,l))$ where $p \in \mathcal{C}$ and l runs through all possible labelling of p with letters $a_1, a_2, \dots$.

Denote by $\mathcal{P}(X)$ the power set of a set X. We consider the commutative diagram

$$\begin{array}{ccc} \mathcal{P}(\mathbb{F}_L) & \xrightarrow{\tilde{\pi}} & \mathcal{P}(G) \\ w \uparrow & & w' \uparrow \\ \mathcal{P}(P) & \xrightarrow{R} & \mathcal{P}(P_{sl}), \end{array}$$

where the maps are given as follows:

- The map $\tilde{\pi}$ is induced by the group homomorphism $\pi : \mathbb{F}_L \to G$, so $\tilde{\pi}(A) := \{\pi(x) \in G | x \in A\}$ for $A \subset \mathbb{F}_L$.
- The map w is given by the above labelling of a partition with any possible choice of letters in $\mathbb{F}_L$, thus for a subset $A \subset P$ of partitions, we have

$$w(A) = \{w(p,l) \in \mathbb{F}_L | p \in A, l = (a_{i(1)}, a_{i(2)}, \dots, a_{i(n)}) \text{ a labelling with pairwise different letters}\}.$$

  Note that we only use the generators a_i of $\mathbb{F}_L$ as letters and not their inverses a_i^{-1}.
- The map R is given by simplification of partitions. To a partition $p \in P$, we assign its simplified partition $p'' \in P_{sl}$, which is possibly the empty partition. Hence

$$R(A) = \{p'' \in P_{sl} \mid p'' \text{ is the simplified partition of a partition } p \in A\}.$$

  If $\mathcal{C}$ is a simplifiable category, then $R(\mathcal{C}) = \mathcal{C}_{sl}$.
- The map w' is given by the labelling of partitions p in single leg form with any possible choice of letters in G, analogous to the map w.

We observe, that the procedure R of simplifying partitions to single leg partitions corresponds to the group homomorphism π, resp. to $\tilde{\pi}$. Furthermore, if $\mathcal{C}$ is a simplifiable category and $\pi(w(p,l)) \neq e$ is an element in $\tilde{\pi} \circ w(\mathcal{C})$ for some partition $p \in \mathcal{C}$ with some labelling l, we may always assume that p is in single leg form. (See Lemma 6.3.1)

We are going to study the structure of $(\tilde{\pi} \circ w)(\mathcal{C})$ for a simplifiable category of partitions $\mathcal{C}$. For this, we translate the category operations to operations in $\mathbb{F}_L$ resp. in G.

Lemma 6.3.3. *Let $\mathcal{C}$ be a simplifiable category of partitions. Then:*

(i) If the word $g = b_1 \dots b_n$ is in $w(\mathcal{C})$, then the reverse word $g' = b_n \dots b_1$ is in $w(\mathcal{C})$.

(ii) If $g, h \in w(\mathcal{C})$, then $gh \in w(\mathcal{C})$.

(iii) $(\tilde{\pi} \circ w)(\mathcal{C}) \subset G$ is a subgroup of G.

Proof. (i) Let $g = w(p,l)$ for some partition $p \in \mathcal{C}$ and some labelling l. Thus, $w(p,l)$ is the word given by labelling the partition p starting in the top left corner and going around clockwise. Since $\mathcal{C}$ is a category, the partition p^* is in $\mathcal{C}$, given by turning p upside down. Labelling p^* with the letters from l starting in the lower left corner and going counterclockwise yields the reverse word $g' = w(p^*, l^*)$. Here, $l^* = (a_{i(n)}, \dots, a_{i(1)})$ denotes the labelling of a partition in an order reverse to the one of the labelling l. Thus, $g' \in w(\mathcal{C})$.

(ii) Let $g = w(p, l)$ and $h = w(q, l')$, where $p, q \in \mathcal{C}$, $l = (a_{i(1)}, \dots, a_{i(n)})$, and $l' = (a_{j(1)}, \dots a_{j(m)})$. By rotation, we may assume that p and q are partitions with no lower points. If all letters of l and l' are pairwise different, then $gh = w(p \otimes q, ll')$, where ll' is the labelling $ll' = (a_{i(1)}, \dots, a_{i(n)}, a_{j(1)}, \dots a_{j(m)})$. Otherwise, denote by M the set of all pairs (α, β) in $\{1, \dots n\} \times \{1, \dots m\}$ such that $i(\alpha) = j(\beta)$. Then gh is obtained from the labelled partition that is constructed by the following:

- Consider the tensor product $p \otimes q$,
- label this partition with the letters $a_{i(1)}, \dots a_{i(k)}, a_{j(1)}, \dots, a_{j(l)}$,
- now for every $(\alpha, \beta) \in M$ join the α-th block of p with the β-th block of q.

The resulting partition r is in $\mathcal{C}$ (by Lemma 6.2.6(b)) and $gh = w(r, l'')$ with the above labelling l''.

(iii) If $\pi(g) \in \tilde{\pi} \circ w(\mathcal{C})$ for $g \in w(\mathcal{C})$, then $\pi(g)^{-1} = \pi(g') \in \tilde{\pi} \circ w(\mathcal{C})$ by (i). By (ii) $\tilde{\pi} \circ w(\mathcal{C})$ is closed under taking products. $\square$

Definition 6.3.4. We denote by F the restriction of $\tilde{\pi} \circ w$ to the set of all simplifiable categories of partitions as a map with image in the subgroups of G.

This map F transfers the problem of classifying the simplifiable categories of partitions to a problem in group theory.

6.3.1 The correspondence between simplifiable categories and subgroups of $\mathbb{Z}_2^{*\infty}$

We will give a description of the image of F in terms of subgroups of $G = \mathbb{Z}_2^{*L}$ that are invariant under certain endomorphisms. This is the content of Theorem 6.3.10. Let us prepare its formulation.

Definition 6.3.5. Let S_0 be the subsemigroup of $\mathrm{End}(G)$ generated by the following endomorphisms.

1. Finite identifications of letters, i.e. for any $n \in \mathbb{N}$ and any choice of indices $i(1), \dots, i(n)$ the map

$$\begin{cases} a_k \mapsto a_{i(k)} & 1 \leqslant k \leqslant n\,, \\ a_k \mapsto a_k & k > n\,. \end{cases}$$

2. Conjugation by any letter a_k, i.e. the map $w \mapsto a_k \cdot w \cdot a_k$.

Definition 6.3.6. Denote by E the subgroup of G consisting of all words of even length.

Proper S_0-invariant subgroups of G and E are described in the following lemma.

Lemma 6.3.7. *E is the unique maximal proper S_0-invariant subgroup of G. Furthermore, every proper S_0-invariant subgroup of E contains only words in which every letter $a_1, a_2, \dots$ appears not at all or at least twice. (Note that in E itself, there are words where a letter appears only once.)*

Proof. Firstly, note that E has index 2 in G, so it is a maximal proper subgroup of G. Secondly, it is S_0-invariant. Now, if an S_0-invariant subgroup $H \leqslant G$ contains a word with an odd number of letters, say $2n+1$, we may use the identification of letters from Definition 6.3.5(i) in order to obtain $a_1 = a_1^{2n+1} \in H$. With $a_1 \in H$, it follows that $a_i \in H$ for all i and hence $H = G$.

Let $H \leqslant E$ be an S_0-invariant subgroup and assume that there exists an element $w \in H$ where w contains a letter a_i only once. Using identification of letters, we may assume that $i = 1$ and all other letters are the same, say a_2. We obtain $a_1 a_2 \in H$ or $a_2 a_1 \in H$, thus $a_i a_j \in H$ for all i, j. Now let $w \in E$ be arbitrary. We can write

$$w = a_{i(1)} a_{i(2)} \cdots a_{i(2n)} = \big(a_{i(1)} a_{i(2)}\big)\big(a_{i(3)} a_{i(4)}\big) \cdots \big(a_{i(2n-1)} a_{i(2n)}\big) \in H\,,$$

for some indices $i(1), i(2), \dots, i(2n)$. So $H = E$ and we have finished the proof. ☐

Lemma 6.3.8. *For any simplifiable category of partitions $\mathcal{C}$, $F(\mathcal{C})$ is a proper S_0-invariant subgroup of E.*

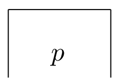
So F is a well-defined map from simplifiable categories of partitions to proper S_0-invariant subgroups of E. Moreover, F is a lattice homomorphism.

Proof. By Lemma 6.3.3 $F(\mathcal{C})$ is a subgroup of G. Since all partitions in $\mathcal{C}$ are of even length, $F(\mathcal{C})$ is a subgroup of E.

Assume that $F(\mathcal{C}) = E$. Then $F(\mathcal{C})$ contains an element in which some letter appears only once. Hence, in $\mathcal{C}$ there is a partition with a singleton. Thus $\mathcal{C}$ is not hyperoctahedral, which is a contradiction. We have shown that $F(\mathcal{C})$ is a proper subgroup of E.

We show that $F(\mathcal{C})$ is invariant under the generating endomorphisms of S_0 in Definition 6.3.5. Let $g = \pi(w(p, l))$ be an element in $F(\mathcal{C})$ constructed from a partition $p \in \mathcal{C}$. It is clear that we can change a letter in g, if the new letter did not appear in g before – this simply corresponds to $\pi(w(p, l'))$ with a different labelling l'. If the new letter already appeared in g, we connect two blocks of p using Lemma 6.2.6. This shows that $F(\mathcal{C})$ is closed under identification of letters.

Furthermore, $F(\mathcal{C})$ is closed under conjugation with a letter a_k. Indeed, let $e \neq g = \pi(w(p, l)) = a_{i(1)} \dots a_{i(m)}$ be an element in $F(\mathcal{C})$. Assume that p is a partition in single leg form with no lower points (see Lemma 6.3.1). If the letter a_k does not appear in the word $a_{i(1)} \dots a_{i(m)}$, we consider the partition

$p' =$  ,

i.e. the partition obtained from p by nesting it into a pair partition ⊓. Labelling this partition with $l' = (a_k, a_{j(1)}, \dots, a_{j(n)})$ for $l = (a_{j(1)}, \dots, a_{j(n)})$ yields $a_k g a_k = \pi(w(p', l'))$ in $F(\mathcal{C})$. On the other hand, if the letter a_k appears in the word $a_{i(1)} \dots a_{i(m)}$, we have four cases.

- If $i(1) \neq k$ and $i(m) \neq k$, we connect the outer pair partition of p' with the block of p which corresponds to the letter a_k (see Lemma 6.2.6). The resulting partition p'' yields $a_k g a_k = \pi(w(p'', l''))$ in $F(\mathcal{C})$ for a suitable labelling l''.

- If $i(1) \neq k$ and $i(m) = k$, the element $a_k g a_k$ is given by $a_k g a_k = a_{i(m)} a_{i(1)} \dots a_{i(m-1)}$ (as $a_k^2 = e$). Therefore, we consider the labelled partition p in a rotated version, which yields $a_k g a_k \in F(\mathcal{C})$. Likewise in the case $i(1) = k$ and $i(m) \neq k$.

- If $i(1) = k$ and $i(m) = k$, the element $a_k g a_k$ equals $a_{i(2)} \dots a_{i(m-1)}$. On the other hand, the very left point and the very right point of p belong to the same block, so, rotating one of them next to the other and erasing them using the pair partition yields a partition $p'' \in \mathcal{C}$ such that $a_k g a_k = \pi(w(p'', l''))$ for a suitable labelling l''.

Finally note that, since $\tilde{\pi} \circ w$ preserves inclusions, its restriction F is a lattice homomorphism. $\square$

The preceding lemma specifies that we can associate an S_0-invariant subgroup $F(\mathcal{C})$ of E to any simplifiable category $\mathcal{C}$ of partitions – but we can also go back. In fact, *every* proper S_0-invariant subgroup of E comes from such a category. This is worked out in the sequel.

Lemma 6.3.9. *For any proper S_0-invariant subgroup H of E, the set*

$$\begin{aligned}\mathcal{C}_H &:= w^{-1}(\pi^{-1}(H)) \\ &= \{p \in P \mid \text{there is a labelling } l \text{ such that } \pi(w(p,l)) \in H\} \subset P\end{aligned}$$

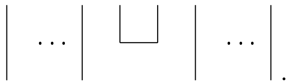
is a simplifiable category of partitions.

Proof. The pair partition $\sqcap$, the unit partition $|$, the four block partition $\sqcap\!\!\sqcap\!\!\sqcap$, and the pair positioner partition $\sqcup\!\!\!/\!\sqcap$ are all in $\mathcal{C}_H$, since they are mapped to the neutral element $e \in H$ for any labelling l.

Let p and q be partitions in $\mathcal{C}_H$ and denote by $g := \pi(w(p,l))$ and $h := \pi(w(q,l'))$ some corresponding elements in H for some labellings l and l'. Since H is invariant under permutation of letters we can assume that the labellings l and l' are such that g and h do not share any letter. The element g may be written as $g = g_1 g_2$, where g_1 corresponds to the labelling of the upper points of p, and g_2 to the lower points of p. Consider the tensor product $p \otimes q$ of p and q as labelled partitions, i.e. we form $p \otimes q$ and label it by a labelling l'' such that the subpartition p in $p \otimes q$ is labelled by l and the subpartition q is labelled by l'. Then, the element $\pi(w(p \otimes q, l''))$ is of the form $g_1 h g_2$. (Recall that the labelling procedure starts at the upper left point of a partition and goes around clockwise – thus, in $p \otimes q$ the upper points of p are labelled first, then the whole of q is labelled, and we finish by labelling the lower points of p.) As H is closed under conjugation, the element $g_1 h g_1^{-1}$ is in H, so is $g_1 h g_2 = g_1 h g_1^{-1} g$. Hence, $p \otimes q \in \mathcal{C}_H$, and $\mathcal{C}_H$ is closed under tensor products.

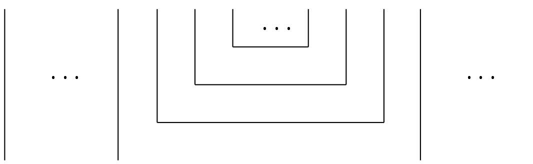
The set $\mathcal{C}_H$ is also closed under involution, since for $p \in \mathcal{C}_H$ with $\pi(w(p,l)) = g \in H$, we have $\pi(w(p^*, l^*)) = g^{-1} \in H$, where l^* denotes the labelling l in reverse order. It is also closed under rotation, since moving points (from above to below or the converse) at the right hand side of a partition p does not change the labelling – and hence $\pi(w(p,l))$ is invariant under this operation. Moving points at the left hand side of p is reflected by conjugating $\pi(w(p,l))$ with the first respectively with the last letter of this word.

It remains to show that $\mathcal{C}_H$ is closed under the composition of partitions. We first show that $\mathcal{C}_H$ is closed under composition with a partition of the form



Let $p \in \mathcal{C}_H$ be a partition on k upper points and m lower points and consider the partition $|| \cdots |\sqcup|| \cdots |$ on m upper points and $m-2$ lower points, where $\sqcup$ connects the i-th and the $(i+1)$-st point. Denote their composition by p'. There is a labelling l such that $g := \pi(w(p,l))$ is in H. For a suitable labelling l', the element $\pi(w(p',l'))$ arises from g by identifying the $(k+i)$-th and the $(k+i+1)$-st letter. Since H is invariant under this operation, the partition p' is in $\mathcal{C}_H$.

It remains to reduce the composition of arbitrary partitions to the previous case. Let $p \in \mathcal{C}_H$ be a partition on k upper and l lower points, and let $q \in \mathcal{C}_H$ be on l upper and m lower points. Write p' and q' for the partitions arising from p and q, respectively, by rotating their lower points to the right of the upper points. Then p' and q' are both in $\mathcal{C}_H$. Composing $p' \otimes q'$ with the partition



k strings on the left, l pair partitions nested into each other, and m strings on the right

yields a partition $p'' \in \mathcal{C}_H$ on $k+m$ points. Rotating m points on the right of p'' to below gives the composition pq of p and q, which hence is in $\mathcal{C}_H$.

We conclude that $\mathcal{C}_H$ is closed under the category operations, hence it is a category of partitions, containing $\sqcup\!\!/\!\!\sqcap$. On the other hand, the partition $\uparrow \otimes \uparrow$ is not in $\mathcal{C}_H$, since $\pi(w(\uparrow \otimes \uparrow, l))$ is a word of the form ab, where a and b are different letters in G. By Lemma 6.3.7, these elements are not in H. Thus, $\mathcal{C}_H$ is simplifiable. □

We show now that the map $H \mapsto \mathcal{C}_H$ is the inverse of F.

Theorem 6.3.10. *The maps F and $H \mapsto \mathcal{C}_H$ are inverse to each other. Hence, the map F is bijective as a map from simplifiable categories of partitions onto proper S_0-invariant subgroups of E.*

Proof. Firstly, let H be a proper S_0-invariant subgroup of E, and let $x \in H$. Denote by p the partition connecting the letters of the word x if and only if they coincide, and let l be the labelling such that $\pi(w(p,l)) = x$. Thus, $p \in \mathcal{C}_H$ and hence $x \in F(\mathcal{C}_H)$. (Recall that $x \in F(\mathcal{C})$ if and only if $x = \pi(w(p,l))$ for some $p \in \mathcal{C}$ and some labelling l.) Conversely, let $x = \pi(w(p,l)) \in F(\mathcal{C}_H)$ where $p \in \mathcal{C}_H$. By definition, there is a labelling l' such that $\pi(w(p,l')) \in H$. Now, H is invariant under exchange of letters, thus $x = \pi(w(p,l)) \in H$. We deduce that $H = F(\mathcal{C}_H)$.

Secondly, let $\mathcal{C}$ be a simplifiable category of partitions, and let $p \in \mathcal{C}$. Then $\pi(w(p,l)) \in F(\mathcal{C})$ for any labelling l, and hence $p \in \mathcal{C}_{F(\mathcal{C})}$. On the other hand, for $p \in \mathcal{C}_{F(\mathcal{C})}$ there is a labelling l such that $\pi(w(p,l)) \in F(\mathcal{C})$. Thus, $\pi(w(p,l)) = \pi(w(q,l'))$ for some partition $q \in \mathcal{C}$ and some labelling l'. By Lemma 6.3.1 and Lemma 6.2.11, we have $p \in \mathcal{C}$. This finishes the proof of $\mathcal{C} = \mathcal{C}_{F(\mathcal{C})}$. □

6.4 Classification and structural results for easy quantum groups

In this section we deduce from Theorem 6.3.10 that there are uncountably many different simplifiable categories. We end this section by giving structural results on the lattice of simplifiable quantum groups.

6.4.1 The classification of simplifiable categories by invariant subgroups of $\mathbb{F}_\infty$

We can identify $E \leqslant \mathbb{Z}_2^{*\infty}$ with a free group and describe the restriction of endomorphisms from S_0 to E. This is the content of the next lemma.

Lemma 6.4.1. *We put $x_k := a_1 a_{k+1}$ for $k = 1, 2, \ldots$ Then $x_1, x_2, \ldots$ is a free basis of E. The restriction $\{\phi_{|E} | \phi \in S_0 \subset \mathrm{End}(G)\}$ of endomorphisms to E is the semigroup generated by*

1. *finite identifications of letters, i.e. for any $n \in \mathbb{N}$ and any choice of indices $i(1), \ldots, i(n)$ the map*

$$\begin{cases} x_k \mapsto x_{i(k)} & 1 \leqslant k \leqslant n\,, \\ x_k \mapsto x_k & k > n\,, \end{cases}$$

2. *for all $i \in \mathbb{N}$, the map defined by $x_k \mapsto x_i^{-1} \cdot x_k$ for all $k \in \mathbb{N}$,*

3. *for all* $k \in \mathbb{N}$, *the map* $x_k \mapsto e$ *that leaves all other letters invariant,*
4. *the map* $x_k \mapsto x_k^{-1}$ *for all* $k \in \mathbb{N}$,
5. *all inner automorphisms of* E,

Proof. We have $x_k^{-1} = a_{k+1}a_1$ for all $k \in \mathbb{N}$. So, every word of even length in G can be written uniquely as a product of the elements $x_1, x_1^{-1}, x_2, x_2^{-1}, \ldots$, so $x_1, x_2, \ldots$ is a free basis for E.

We check that the endomorphisms (i)-(iv) are precisely the restrictions of the generators of S_0 as given in Definition 6.3.5. In order to obtain (i) it suffices to consider the endomorphism of G defined by $a_{k+1} \mapsto a_{i(k)+1}$ for $1 \leqslant k \leqslant n$ and leaving all other letters invariant. For (ii), we have to consider the endomorphism of G mapping $a_1 \mapsto a_{i+1}$ and leaving all other letters invariant. For (iii), we have to take the map $a_{k+1} \mapsto a_1$. We considered all possible endomorphisms from item (i) in Definition 6.3.5. The endomorphisms in (iv) are obtained by mapping $a_{k+1} \mapsto a_1 a_{k+1} a_1$ and for all $k \in \mathbb{N}$. The conjugation by x_i is obtained by $a_k \mapsto a_{i+1} a_k a_{i+1}$ for all $k \in \mathbb{N}$ composed with the endomorphism in (iv). Indeed $x_k = a_1 a_{k+1}$ is mapped to $a_{i+1} a_1 a_{k+1} a_{i+1} = x_i^{-1} x_k^{-1} x_i$ and the endomorphism in (iv) maps this element to $x_i x_k x_i^{-1}$. We considered all possible endomorphisms from both items (i) and (ii) in Definition 6.3.5, so the restriction of S_0 to E is equal to the semigroup described in the statement. ☐

Note that the endomorphisms defined in the previous lemma depend on the choice of the free basis $x_1, x_2, \ldots$. Fixing this choice, we obtain an isomorphism $E \cong \mathbb{F}_\infty$.

Definition 6.4.2. We denote by S the subsemigroup of $\mathrm{End}(\mathbb{F}_\infty)$ generated by the maps in Lemma 6.4.1.

Lemma 6.4.3. *A subgroup of* $\mathbb{F}_\infty$ *is* S*-invariant, if and only if*

1. *it is closed under identification and deletion of letters,*
2. *for all* $i \in \mathbb{N}$ *it is closed under the map* $x_k \mapsto x_i^{-1} \cdot x_k$ *for all* $k \in \mathbb{N}$,
3. *it is closed under the map* $x_k \mapsto x_k^{-1}$ *for all* $k \in \mathbb{N}$,
4. *and it is normal.*

Proof. This is a translation of Lemma 6.4.1. Item (i) and (iii) of Lemma 6.4.1 correspond to item (i) here. The items (ii) correspond to each other, item (iv) of Lemma 6.4.1 corresponds to item (iii) here and normality of an S-invariant subgroup is the same as invariance under inner automorphism (item (v) of Lemma 6.4.1). ☐

Theorem 6.3.10 can be now translated into the more convenient setting of subgroups of $\mathbb{F}_\infty$.

Theorem 6.4.4. *The map F of Definition 6.3.4 induces a lattice isomorphism between simplifiable categories of partitions and proper S-invariant subgroups of $\mathbb{F}_\infty$.*

Proof. This follows from Theorem 6.3.10 and Lemma 6.4.1 □

The formulation of the previous theorem allows us to employ a well-known subset of the S-invariant subgroups of $\mathbb{F}_\infty$. The next observation is essential for the rest of this section.

Remark 6.4.5. Every fully characteristic subgroup (see Section 6.1.4 for a definition) of $\mathbb{F}_\infty$ is S-invariant. Hence F induces a lattice embedding of proper fully characteristic subgroups of $\mathbb{F}_\infty$ into simplifiable categories of partitions.

To close this section, let us ask whether or not also the other implication holds: Is every S-invariant subgroup of $\mathbb{F}_\infty$ fully characteristic? We only have a partial answer to this question.

Proposition 6.4.6. *Every S-invariant subgroup of $\mathbb{F}_\infty$ that contains the commutator $x_1x_2x_1^{-1}x_2^{-1}$ is fully characteristic.*

Proof. It suffices to show that the map sending an S-invariant subgroup of $\mathbb{F}_\infty$ to its fully characteristic closure is injective on subgroups containing $x_1x_2x_1^{-1}x_2^{-1}$. A proper S-invariant subgroup $H \leqslant \mathbb{F}_\infty$ contains the commutator $x_1x_2^{-1}x_1^{-1}x_2 = a_1a_2a_3a_1a_2a_3$ if and only if the associated category of partitions contains the half-liberating partition. Similarly, the s-mixing partition h_s corresponds to the element $a_1a_2a_1a_2 \ldots a_1a_2$ (s repetitions), which is equal to x_1^s. So by [232, Theorem 4.13], it suffices to prove that the fully characteristic subgroups generated by $x_1x_2x_1^{-1}x_2^{-1}$ and x_1^s are pairwise different for different $s \in \mathbb{N}\backslash\{1\}$. By the fact that the group $\mathbb{Z}/s\mathbb{Z}$ is abelian and has exponent s but not exponent s' for $s' < s$, invoking the correspondence between fully characteristic subgroups of $\mathbb{F}_\infty$ and varieties of groups from Theorem 6.1.8, we finish the proof. □

6.4.2 Classification results for easy quantum groups

The link between the theory of varieties of groups and easy quantum groups is given by Theorem 6.4.4 and Remark 6.4.5. Let us state this more precisely.

Theorem 6.4.7. *There is a lattice injection from the lattice of non-empty varieties of groups into the lattice of simplifiable easy quantum groups.*

Proof. The lattice of simplifiable quantum groups is anti-isomorphic to the lattice of simplifiable categories of partitions. Theorem 6.4.4 shows that the latter lattice is isomorphic to the lattice of proper S-invariant subgroups of $\mathbb{F}_\infty$. By Remark 6.4.5, there is an injection of lattices of proper fully characteristic subgroups of $\mathbb{F}_\infty$ into the lattice of proper S-invariant subgroups of $\mathbb{F}_\infty$. The former is anti-isomorphic with the lattice of non-empty varieties of groups by Theorem 6.1.8. Composing these isomorphisms, injections and anti-isomorphisms, we obtain an injection of lattices as in the statement of the theorem. □

Remark 6.4.8. The proof of the previous theorem also shows that there is a one-to-one correspondence between varieties of groups and certain simplifiable categories of partitions. We hence obtain a combinatorial and a quantum group perspective on varieties of groups.

The correspondence from the last theorem allows us to translate known results about varieties of groups into statements about easy quantum groups. Let us start with some results about the classification of easy quantum groups.

In [17], the question was raised whether or not all easy quantum groups are either classical, free, half-liberated or form part of a multi-parameter family unifying the series of quantum groups $H_n^{(s)}$ and $H_n^{[s]}$. We can answer this question in the negative.

Theorem 6.4.9. *There are uncountably many pairwise non-isomorphic easy quantum groups.*

This follows directly from Theorem 6.4.7 and the following result by Olshanskii.

Theorem 6.4.10 (See [148])**.** *The class of varieties of groups has cardinality equal to the continuum.*

Easy quantum groups offer a class of examples, which is concretely accessible by means of combinatorics. Therefore, it would be good to amend Theorem 6.4.9 with concrete examples. Unfortunately, partitions are not well-suited to represent higher commutators in E. We therefore omit a concrete translation of the following result of Vaughan-Lee. The notation $[x_1, x_2, x_3, \dots, x_n]$ denotes the higher commutator $[[[\cdots[[x_1, x_2], x_3], x_4], \dots], x_n]$.

Theorem 6.4.11 (See [221])**.** *Let $x, y, z, x_1, x_2, \dots$ be a free basis of $\mathbb{F}_\infty$. Denote $w_k = [[x,y,z],[x_1,x_2],[x_3,x_4],\dots,[x_{2k-1},x_{2k}],[x,y,z]]$. Then the fully characteristic subgroups of $\mathbb{F}_\infty$ generated by*

$$\{w_k \mid k \in I\} \cup \{x^{16}, [[x_1,x_2,x_3],[x_4,x_5,x_6],[x_7,x_8]]\}\,, \quad I \subset \mathbb{N}$$

are pairwise different.

It would be interesting to find an uncountable family of categories of partitions, which is more natural from the point of view of combinatorics.

Remark 6.4.12. Given a certain class of objects that we want to classify up to a given equivalence relation $\simeq$, it is generally known that the cardinality of the quotient set does not say a lot about the difficulty of the classification. The difference between the classification of torsion free abelian groups of rank 1 [6] and of rank $\geqslant 2$ [130, 135, 204] is a classical instance of this fact. The theory of *Borel reducibility* offers a better point of view on classification problems of this kind. See [128] for an exposition. If $\mathcal{R}, \mathcal{S}$ are equivalence relations on Polish spaces X and Y, respectively, then $\mathcal{R}$ is called Borel reducible to $\mathcal{S}$, if there is a Borel map $f : X \to Y$ such that $x_1 \simeq_{\mathcal{R}} x_2 \Leftrightarrow f(x_1) \simeq_{\mathcal{S}} f(x_2)$. In common terms, $\mathcal{R}$ is "easier" then $\mathcal{S}$. We call $\mathcal{R}$ *smooth*, if it is reducible to the equivalence relation of equality of points on some Polish space Y.

Denoting by P the set of all partitions, the space $X = 2^P$ of subsets of P is a Polish space. Denote by $\mathcal{R}_{QG}$ the equivalence relation on X making $x_1, x_2 \in X$ equivalent, if and only if they generate the same category of partitions, i.e. $\langle x_1 \rangle = \langle x_2 \rangle$. We show that $\mathcal{R}_{QG}$ is smooth. From the point of view of Borel complexity, this can be interpreted as saying that it is comparably easy to decide whether two subsets of partitions generate the same category of partitions. However, the lattice of easy quantum groups is not traceable, as is demonstrated by Theorem 6.4.7. We thank Simon Thomas for pointing out the following argument to us. The set $\mathcal{CAT} \subset X$ of categories of partitions is a Borel subset and hence the Borel space $\mathcal{CAT}$ is isomorphic to the space of Borel sets on some Polish space. Moreover, the map gen : $X \to \mathcal{CAT}$ sending $x \in X$ to the category of partitions that it generates is Borel. It follows that $\mathcal{R}_{QG}$ is smooth.

For the sake of completeness, let us elaborate on the above argument. We show that $\mathcal{CAT} \subset X$ is Borel and that gen is a Borel map. Consider the following maps.

- Tens : $X \times X \to X$ defined by $\mathrm{Tens}(x, y) = \{p \otimes q \mid p \in x, q \in y\}$.
- Comp : $X \times X \to X$ defined by $\mathrm{Comp}(x, y) = \{pq \mid p \in x$ and $q \in y$ are composable$\}$.
- $* : X \to X : x \mapsto \{p^* \mid p \in x\}$.

Proving that the preimages of the sets $\{x \in X \mid p \in x\}$, where p runs through all partitions, are Borel, one can show that all the maps above are Borel. Note that the rotation operations of a category of partitions can be deduced from its other properties (see Remark 1.6 of [232]). It follows that $\mathcal{CAT} \subset X$ is a Borel set, since it is the intersection of the fixed points of the above maps with the set $\{x \in X \mid x$ contains the identity partition $|$ and the pair partition $\sqcap\}$. It also follows that $x \mapsto \mathrm{gen}(x)$ is Borel, since $\mathrm{gen}(x)$ arises from x as the union of countably many iterated applications of the above maps to $x \cup \{|^{\otimes n} \mid n \in \mathbb{N}\} \cup \{\sqcap\}$.

Remark 6.4.13. If (A, u) is a hyperoctahedral quantum group with associated category of partitions $\mathcal{C}$, then $\uparrow \otimes \uparrow \notin \mathcal{C}$. As a consequence, $\mathrm{Hom}(u, u)$ is one dimensional and hence u is an irreducible corepresentation of A. It follows that the tensor C*-category of unitary finite dimensional corepresentations of (A, u) is generated by a single irreducible element. So Theorem 6.4.9 gives rise to many new tensor C*-categories, whose fusion rules are described by the combinatorics of categories of partitions. It remains an interesting question to determine these fusion rules.

6.4.3 Structural results for easy quantum groups

We position known hyperoctahedral quantum groups in the context of varieties of groups. See also Example 6.1.7.

Example 6.4.14.

1. The variety of all groups corresponds to the trivial subgroup of $\mathbb{F}_\infty$, which in turn corresponds to the maximal simplifiable quantum group $H_n^{[\infty]}$ (resp. to the category $\langle \sqcup\!\!\!/\!\!\sqcap \rangle$).
2. By Proposition 6.4.6, the category $\langle \times\!\!\!\!|\,, \sqcap\!\sqcap\!\sqcap \rangle$ corresponds to the commutator subgroup of $\mathbb{F}_\infty$. So Example 6.1.7(ii) shows that the quantum group H_n^* corresponds to the variety of all abelian groups.
3. By the same Proposition and Examples 6.1.7 (ii) and (iii), the categories $\langle \times\!\!\!\!|\,, \sqcap\!\sqcap\!\sqcap, h_s \rangle$ correspond to the fully characteristic subgroup of $\mathbb{F}_\infty$ generated by the commutator subgroup and x_1^s. It follows that the easy quantum group $H_n^{(s)}$, $s \geqslant 2$ corresponds to the variety of abelian groups of exponent s. Note that $H_n^{(2)} = H_n$ is a group.

We end this section by giving two structural results regarding the classification of simplifiable quantum groups.

For a simplifiable category $\mathcal{C}$ denote by $\mathcal{C}^0$ its intersection with $\langle \times\!\!\!\!|\,, \sqcap\!\sqcap\!\sqcap \rangle$. The following theorem generalises Theorem 4.13 of [232].

Proposition 6.4.15. *Every simplifiable easy quantum group G is either an intermediate quantum subgroup $H_n^+ \supset G \supset H_n^*$ or it is the intersection of some $H_n^+ \supset G_0 \supset H_n^*$ and $H_n^{[s]}$ for some $s \geqslant 2$. Equivalently, for every simplifiable category $\mathcal{C}$ that is not contained in $\langle \times\!\!\!\!|\,, \sqcap\!\sqcap\!\sqcap \rangle$, there is $s \geqslant 2$ such that $\mathcal{C} = \langle \mathcal{C}^0, h_s \rangle$.*

Proof. Take a simplifiable category $\mathcal{C}$ that does not contain the crossing partition. Let $H \leqslant \mathbb{F}_\infty$ be the S-invariant subgroup of $\mathbb{F}_\infty$ associated with $\mathcal{C}$ by Theorem 6.4.4.

Take $w \in H$. The exponent of x_i in w is by definition the sum of the powers of x_i that appear in w. Denote by e_i the exponent of x_i. For all i, we obtain $x_i^{e_i} \in H$, by applying to w the endomorphism of $\mathbb{F}_\infty$ that erases all letters of w except for x_i. For later use, note that since $[\mathbb{F}_\infty, \mathbb{F}_\infty]$ is the kernel of the abelianisation map $\mathbb{F}_\infty \to \mathbb{Z}^\infty$, we can write for some $n \in \mathbb{N}$ and for some word $c \in H \cap [\mathbb{F}_\infty, \mathbb{F}_\infty]$

$$w = x_1^{e_1} \cdots x_n^{e_n} (x_n^{-e_n} \cdots x_1^{-e_1} w) = x_1^{e_1} \cdots x_n^{e_n} c\,.$$

Let s be the minimal number such that $x_1^s \in H$. By the previous decomposition of words we see that H is generated as an S-invariant subgroup by x_1^s and by $H \cap [\mathbb{F}_\infty, \mathbb{F}_\infty]$. Appealing to the correspondence between simplifiable categories of partitions and S-invariant subgroups of $\mathbb{F}_\infty$ in Theorem 6.4.4, we have finished the proof. □

Proposition 6.4.16. *Every simplifiable quantum group $G \neq H_n$ has $H_n^{(s)}$ as a quantum subgroup for some $s \geqslant 3$. Equivalently, every simplifiable category of partitions that does not contain the crossing ╳ is contained in ⟨⚹, ⊓⊓⊓, h_s⟩ for some $s \geqslant 3$.*

Proof. This follows from Proposition 6.4.15: let $G \neq H_n$ be a simplifiable quantum group. Then either G contains H_n^* or it contains $H_n^{(s)} = H_n^* \cap H_n^{[s]}$ for some $s \geqslant 3$. □

6.5 The C*-algebras associated to the simplifiable categories

Given a category of partitions $\mathcal{C}$, we denote by $(A_\mathcal{C}(n), u_n)$ the compact matrix quantum group with fundamental corepresentation of size $n \times n$ associated with $\mathcal{C}$. In this section, we study the C*-algebras associated with the categories $\mathcal{C} =$ ⟨⊔⁄⊓⟩, denoted by $A_\mathcal{C}(n) = \mathrm{C}(H_n^{[\infty]})$. We also study the C*-algebras $A_\mathcal{C}(n)$ for $\mathcal{C} =$ ⟨⊔⊔╳⊓⊓⟩, since some of their theory is similar.

Recall that the hyperoctahedral quantum group H_n^+ corresponds to the category ⟨⊓⊓⊓⟩. If $G \subset O_n^+$ is a compact quantum subgroup of O_n^+ and u denotes the fundamental corepresentation of $\mathrm{C}(G)$, then the map T_p for $p =$ ⊓⊓⊓ is in the intertwiner space $\mathrm{Hom}(1, u^{\otimes 4})$ if and only if $u_{ik}u_{jk} = u_{ki}u_{kj} = 0$ whenever

$i \neq j$. Hence, this relation is fulfilled for all compact quantum subgroups $G \subset H_n^+$. We also have $\sum_k u_{ik}^2 = \sum_k u_{kj}^2 = 1$, as well as $u_{ij} = u_{ij}^*$, for all i, j (see [232] for the relations of $C(H_n^+)$).

Recall also the definition of the linear maps $T_p : (\mathbb{C}^n)^{\otimes k} \to (\mathbb{C}^n)^{\otimes l}$ indexed by a partition $p \in P(k, l)$:

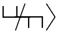
$$T_p(e_{i(1)} \otimes \cdots \otimes e_{i(k)}) = \sum_{j(1),\ldots,j(l)=1}^{n} \delta_p(i,j) \cdot e_{j(1)} \otimes \cdots \otimes e_{j(l)} .$$

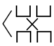
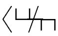
Here, $e_1, \ldots, e_n$ is the canonical basis of $\mathbb{C}^n$, and $\delta_p(i,j) = 1$ if and only if the indices of $i = (i(1), \ldots, i(k))$ and $j = (j(1), \ldots, j(l))$ that are connected by the partition p coincide. Otherwise $\delta_p(i,j) = 0$. (cf. [24, Definitions 1.6 and 1.7])

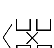
Lemma 6.5.1. *Let $G \subset H_n^+$ be a compact quantum subgroup of H_n^+ and denote by $\mathrm{C}(G)$ its corresponding C^*-algebra generated by the entries of the fundamental corepresentation u_{ij}, $i, j = 1, \ldots, n$. Then*

1. *$T_p \in \mathrm{Hom}(u^{\otimes 4}, u^{\otimes 4})$ for $p =$*  *if and only if $u_{ij}^2 u_{kl}^2 = u_{kl}^2 u_{ij}^2$ for all i, j, k, l.*
2. *$T_p \in \mathrm{Hom}(u^{\otimes 3}, u^{\otimes 3})$ for $p =$*  *if and only if $u_{ij} u_{kl}^2 = u_{kl}^2 u_{ij}$ for all i, j, k, l.*

Proof. Compare $u^{\otimes 4}(T_p \otimes 1)$ with $(T_p \otimes 1)u^{\otimes 4}$ for (i) and analogous for (ii). □

We will now describe the C*-algebras corresponding to the categories $\langle$$\rangle$ and $\langle$$\rangle$.

Proposition 6.5.2. *The C^*-algebras $A_{\mathcal{C}}(n)$ associated with $\langle$$\rangle$ and $\langle$$\rangle$ are universal C^*-algebras generated by elements u_{ij}, $i, j, = 1, \ldots, n$ such that*

1. *the u_{ij} are local symmetries (i.e. $u_{ij} = u_{ij}^*$ and u_{ij}^2 is a projection),*
2. *the projections u_{ij}^2 fulfill $\sum_k u_{ik}^2 = \sum_k u_{kj}^2 = 1$ for all i, j,*
3. *in the case $\mathcal{C} = \langle$$\rangle$, we also have $u_{ij}^2 u_{kl}^2 = u_{kl}^2 u_{ij}^2$ for all i, j, k, l,*
4. *in the case $\mathcal{C} = \langle$$\rangle$, we even have $u_{ij}^2 u_{kl} = u_{kl} u_{ij}^2$ for all i, j, k, l.*

Proof. The C*-algebras $A_{\mathcal{C}}(n)$ fulfill the relations of $C(H_n^+)$, the C*-algebra associated with the free hyperoctahedral quantum group H_n^+. It follows that

u_{ij}^2 is a projection, since $u_{ij}^2 = u_{ij}^2(\sum_k u_{ik}^2) = \sum_k u_{ij}(u_{ij}u_{ik})u_{ik} = u_{ij}^4$. Lemma 6.5.1 shows that in addition the relations (iii) or (iv) hold, respectively.

In order to show, that $A_{\mathcal{C}}(n)$ is universal with the above relations, note that $\mathrm{C}(H_n^+)$ is universal with the relations in (i) and (ii). So $A_{\mathcal{C}}(n)$ is the quotient of this universal C*-algebra, by the relations imposed by Lemma 6.5.1. So it is the universal C*-algebra for the relations (i), (ii) and (iii) or (i), (ii) and (iv), respectively. □

This proposition shows that the elements u_{ij}^2 fulfill the relations of S_n, the (classical) permutation group (or rather of $C(S_n)$). The squares of the elements u_{ij} of the above C*-algebras $A_{\mathcal{C}}(n)$ thus behave like commutative elements, whereas the u_{ij} itself behave like free elements. The quantum groups $A_{\mathcal{C}}(n)$ are hence somewhat in between the commutative and the (purely) non-commutative world.

Remark 6.5.3. From the description of their C*-algebras in Proposition 6.5.2, we can deduce that $\langle \begin{smallmatrix}\sqcup\sqcup\\ \times \\ \sqcap\sqcap\end{smallmatrix} \rangle \neq \langle \sqcup\!\!/\!\!\sqcap \rangle$ by showing that the canonical quotient map from $A_{\mathcal{C}}(n)$ to $A_{\mathcal{C}'}(n)$ for $\mathcal{C} = \langle \begin{smallmatrix}\sqcup\sqcup\\ \times \\ \sqcap\sqcap\end{smallmatrix} \rangle$, $\mathcal{C}' = \langle \sqcup\!\!/\!\!\sqcap \rangle$ is not an isomorphism. Indeed take $n = 3$ and let $H = \mathbb{C}^3$ with its canonical basis e_1, e_2, e_3. Denote by p_i the projection of H onto $\mathbb{C}e_i$, $i \in \{1, 2, 3\}$. We define operators w_{ij}, $1 \leqslant i, j \leqslant 3$ on H, which define a representation of $A_{\mathcal{C}}(n)$ but not of $A_{\mathcal{C}'}(n)$. Let $w_{11}e_1 = e_2$, $w_{11}e_2 = e_1$ and $w_{11}e_3 = 0$ and $w_{22} = p_1$. Then $w_{11}^2 w_{22}^2 = w_{22}^2 w_{11}^2$, but $w_{11}w_{22}^2 \neq w_{22}^2 w_{11}$. Define

$$w_{21} = w_{12} = p_3\,, \quad w_{31} = w_{13} = 0$$

$$w_{23} = w_{32} = p_2\,, \quad w_{33} = p_1 + p_3\,.$$

Then $(w_{ij})_{ij}$ induces a representation of $A_{\mathcal{C}}(n)$, which does not factor through $A_{\mathcal{C}}(n) \to A_{\mathcal{C}'}(n)$.

Remark 6.5.4. 1. If $\pi : \mathrm{C}(H_n^{[\infty]}) \to \mathscr{B}(H)$ is an irreducible representation, the projections $\pi(u_{ij}^2)$ are either 1 or 0, since they commute with all elements of $\pi(\mathrm{C}(H_n^{[\infty]}))$. Since $\sum_k \pi(u_{ik}^2) = \sum_k \pi(u_{ki}^2) = 1$, there is a permutation $\gamma \in S_n$ such that $\pi(u_{ij}^2) = 1$, if $\gamma(i) = j$, and $\pi(u_{ij}^2) = 0$, otherwise. Recall, that the full group C*-algebra $\mathrm{C}^*_{\max}(\mathbb{Z}_2^{*n})$ is isomorphic to the n-fold unital free product of the C*-algebra $\mathrm{C}^*_{\max}(\mathbb{Z}_2) = \mathbb{C}^2$. The latter is the universal C*-algebra generated by a symmetry. Thus, $\mathrm{C}^*_{\max}(\mathbb{Z}_2^{*n})$ is the universal unital C*-algebra generated by n symmetries $w_1, \ldots, w_n$. So, the map $\pi_0(w_i) := \pi(u_{i\gamma(i)})$ defines a representation of $\mathrm{C}^*_{\max}(\mathbb{Z}_2^{*n})$.

2. Vice versa, we can produce representations of $C(H_n^{[\infty]})$ by permutations. First note that the relations of $C(H_n^{[\infty]})$ are invariant under permutation of rows or columns of its fundamental corepresentation. Let $\pi_0 : C^*_{\max}(\mathbb{Z}_2^{*n}) \to \mathscr{B}(H)$ be a representation and let $\gamma \in S_n$ be a permutation. The map $C(H_n^{[\infty]}) \to C^*_{\max}(\mathbb{Z}_2^{*n}) : u_{ij} \mapsto \delta_{j\gamma(i)} \cdot w_i$ is well defined. So $\pi(u_{ij}) := \delta_{j\gamma(i)}\pi_0(w_i)$ defines a (not necessarily irreducible) representation of $C(H_n^{[\infty]})$.

In the C*-algebra associated with a simplifiable category $\mathcal{C}$, the relations on the generators may be read directly from the partitions in single leg form. Let $p = a_{i(1)} \cdots a_{i(k)} \in P(0,k)$ be a partition without upper points in single leg form. We consider p as a word in the letters $a_1, \ldots, a_m$ (labelled from left to right). If we replace the letters a_i, $1 \leqslant i \leqslant m$, in p by some choice of generators u_{ij}, $1 \leqslant i,j \leqslant n$, we obtain an element $a_{i(1)} \cdots a_{i(k)} \in A_{\mathcal{C}}(n)$; replacing the letters by the according elements u_{ij}^2 yields a projection $q \in A_{\mathcal{C}}(n)$.

Proposition 6.5.5. *Let $\mathcal{C}$ be a simplifiable category and let $p = a_{i(1)} \cdots a_{i(k)}$ be a partition in single leg form. The following assertions are equivalent:*

1. *$p \in \mathcal{C}$.*
2. *$a_{i(1)} \cdots a_{i(k)} = q$ in $A_{\mathcal{C}}(n)$ for all choices $a_r \in \{u_{ij} \mid i,j = 1,\ldots,n\}$, $1 \leqslant r \leqslant m$, where q is the according range projection.*
3. *For some $1 \leqslant s \leqslant k$, we have $qa_{i(1)} \cdots a_{i(s)} = qa_{i(k)} \cdots a_{i(s+1)}$ in $A_{\mathcal{C}}(n)$ for all choices $a_r \in \{u_{ij} \mid i,j = 1,\ldots,n\}$, $1 \leqslant r \leqslant m$, where q is the according range projection.*

Proof. The linear map $T_p : \mathbb{C} \to (\mathbb{C}^n)^{\otimes k}$ associated with p is given by

$$T_p(1) = \sum_{i(1),\ldots,i(k)=1}^{n} \delta_p(i) e_{i(1)} \otimes \cdots \otimes e_{i(k)} .$$

We have

$$u^{\otimes k}(T_p \otimes 1)(1 \otimes 1) =$$

$$\sum_{i(1),\ldots,i(k)=1}^{n} e_{i(1)} \otimes \cdots \otimes e_{i(k)} \otimes \left(\sum_{j(1),\ldots,j(k)=1}^{n} \delta_p(j) \cdot u_{i(1)j(1)} \cdots u_{i(k)j(k)} \right),$$

so that $p \in \mathcal{C}$, if and only if for all multi-indices $i = (i(1), \ldots, i(k))$ the equation:

$$\sum_{j(1),\ldots,j(k)=1}^{n} \delta_p(j) \cdot u_{i(1)j(1)} \cdots u_{i(k)j(k)} = \delta_p(i)$$

holds. Now, assume (i) and let us show (ii). Make a choice of $a_r \in \{u_{ij} \mid i, j = 1, \ldots, n\}$ for all $r \in \{1, \ldots, m\}$. Then there are multi-indices i and j satisfying $\delta_p(i) = \delta_p(j) = 1$ such that $a_{i(1)} \cdots a_{i(k)} = u_{i(1)j(1)} \cdots u_{i(k)j(k)}$. Let q be the projection given by $q := u^2_{i(1)j(1)} \cdots u^2_{i(k)j(k)}$. Then

$$\begin{aligned} u_{i(1)j(1)} \cdots u_{i(k)j(k)} &= u_{i(1)j(1)} \cdots u_{i(k)j(k)} \\ &\qquad \left(\sum_{r(1),\ldots,r(k)=1}^{n} \delta_p(r) \cdot u_{i(k)r(k)} \cdots u_{i(1)r(1)} \right) \\ &= \sum_{r(1),\ldots,r(k)=1}^{n} \delta_p(r) \cdot u_{i(1)j(1)} \cdots u_{i(k-1)j(k-1)} \\ &\qquad (\delta_{j(k)r(k)} u^2_{i(k)j(k)}) u_{i(k-1)r(k-1)} \cdots u_{i(1)r(1)} \\ &= \sum_{r(1),\ldots,r(k)=1}^{n} \delta_p(r) \cdot \delta_{i(1)r(1)} u^2_{i(1)j(1)} \cdots \delta_{i(k)r(k)} u^2_{i(k)j(k)} \\ &= q\,. \end{aligned}$$

This proves (ii). Conversely, assume (ii) and let i be any multi-index. If $\delta_p(i) = 0$, then $u_{i(1)j(1)} \cdots u_{i(k)j(k)} = 0$ for any multi-index j that satisfies $\delta_p(j) = 1$, since in this product there are at least two local symmetries that have mutually orthogonal support in the centre of $A_{\mathcal{C}}(n)$. Hence

$$\sum_{j(1),\ldots,j(k)=1}^{n} \delta_p(j) \cdot u_{i(1)j(1)} \cdots u_{i(k)j(k)} = 0\,.$$

Similarly, using the assumption (ii), if $\delta_p(i) = 1$, then

$$u^2_{i(1)j(1)} \cdots u^2_{i(k)j(k)} = \begin{cases} u_{i(1)j(1)} \cdots u_{i(k)j(k)}\,, & \text{if } \delta_p(j) = 1\,, \\ 0\,, & \text{otherwise.} \end{cases}$$

We obtain that

$$\sum_{j(1),\ldots,j(k)=1}^{n} \delta_p(j) \cdot u_{i(1)j(1)} \cdots u_{i(k)j(k)} = \sum_{j(1),\ldots,j(k)=1}^{n} u^2_{i(1)j(1)} \cdots u^2_{i(k)j(k)} = 1\,.$$

This proves (i). The assertions (ii) and (iii) are equivalent, since all projections a_r^2, $1 \leqslant r \leqslant m$ are absorbed by q and $qa_{i(1)} \cdots a_{i(k)} = a_{i(1)} \cdots a_{i(k)}$. $\square$

Let us recall the notion of a coopposite quantum group. If (A, Δ) is a compact quantum group, then its coopposite version is the quantum group $(A, \Sigma \circ \Delta)$, where $\Sigma : A \otimes A \to A \otimes A$ is the flip. In particular, if (A, u) is a compact matrix quantum group, its coopposite version is the compact matrix quantum group (A, u^t).

Corollary 6.5.6. *Every easy quantum subgroup of* $(\mathrm{C}(H_n^{[\infty]}), u_{\mathrm{simpl}})$ *is isomorphic to its coopposite version.*

Proof. Let $(A_{\mathcal{C}}(n), u)$ be an easy quantum subgroup of $\mathrm{C}(H_n^{[\infty]}, u_{\mathrm{simpl}})$ with corresponding category of partitions $\mathcal{C}$. By Proposition 6.5.5, $(A_{\mathcal{C}}(n), u)$ is the universal C*-algebra such that

- all u_{ij} are local symmetries whose support is central in $A_{\mathcal{C}}(n)$ and sums up to 1 in every row and every column
- for any partition $p = a_{i(1)} \cdots a_{i(k)} \in \mathcal{C}$ and any choice of elements $a_r \in \{u_{ij} \mid 1 \leqslant i, j \leqslant n\}$, $1 \leqslant r \leqslant m$ we have that $a_{i(1)} \cdots a_{i(k)}$ is a projection.

These relations are invariant under taking the transpose of u. So $u_{ij} \mapsto u_{ji}$ is a *-automorphism of $A_{\mathcal{C}}(n)$. This finishes the proof. $\square$

Example 6.5.7. Let p be the word $p = abcbcacb$ and consider the C*-algebras associated with the category $\langle \sqcup\!\!\!\!/\!\!\sqcap, p\rangle$. Note that the equation $abcbcacb = 1$ in $\mathbb{Z}_2 * \mathbb{Z}_2 * \mathbb{Z}_2$ is equivalent to $abcb = bcac$. The idea is that Proposition 6.5.5 yields the commutation relations

$$u_{ij}u_{kl}u_{rs}u_{kl} = u_{kl}u_{rs}u_{ij}u_{rs}$$

in $A_{\mathcal{C}}(n)$ with $p \in \mathcal{C}$ "wherever it makes sense". To be more precise, we have

$$qu_{ij}u_{kl}u_{rs}u_{kl} = qu_{kl}u_{rs}u_{ij}u_{rs},$$

where $q = u_{ij}^2 u_{kl}^2 u_{rs}^2$. By multiplying from left or right with the local symmetries u_{ij}, u_{kl}, and u_{rs}, we also obtain relations like for instance

$$qu_{ij}u_{kl}u_{rs}u_{kl}u_{rs}u_{ij} = qu_{kl}u_{rs} \quad \text{or} \quad u_{ij}u_{kl}u_{rs}u_{kl}u_{rs}u_{ij}u_{rs}u_{kl} = q\,.$$

Remark 6.5.8. The K-theory for the C*-algebras associated with easy quantum groups is relatively unknown. Voigt computed the K-theory for O_n^+ in [227] and there are some small extensions of this result to other easy quantum groups in [232]. Let us also mention [222].

6.6 Diagonal subgroups and their quantum isometry groups

If $\mathcal{C}$ is a simplifiable category of partitions, the group $F(\mathcal{C})$ from Theorem 6.3.10 can be recovered directly from $(A_{\mathcal{C}}(n), u_n)_{n\geqslant 2}$. Vice versa, $A_{\mathcal{C}}(n)$ arises as a natural subgroup of the quantum isometry group of $F(\mathcal{C})$. This is explained by the following results.

Definition 6.6.1. For $H \subset \mathbb{Z}_2^{*\infty}$ write

$$(H)_n = \{w \in H \mid w \text{ only involves the letters } a_1, \ldots, a_n\}.$$

Definition 6.6.2. Given a compact matrix quantum group (A, u), its diagonal subgroup $\mathrm{Diag}(A, u)$ is the discrete group that is generated by the image of the diagonal entries of u in the quotient C*-algebra $A/\langle u_{ij} \mid i \neq j\rangle$.

Theorem 6.6.3. *Let $\mathcal{C}$ be a simplifiable category of partitions. Then* $\mathrm{Diag}(A_{\mathcal{C}}(n), u_n) \cong \mathbb{Z}_2^{*n}/F(\mathcal{C})_n$

Proof. Let H be the diagonal subgroup of $A_{\mathcal{C}}(n)$. The C*-algebra $A_{\mathcal{C}}(n)$ is the quotient of $A_o(n)$ by the relations $T_p \otimes 1 = (u_n^o)^{\otimes k}(T_p \otimes 1)$ for all $p \in \mathcal{C}$. Moreover, the diagonal subgroup of the free orthogonal quantum group satisfies $\mathrm{Diag}(A_o(n), u_n^o) = \mathbb{Z}_2^{*n}$. Denote by π the quotient homomorphism $A_{\mathcal{C}}(n) \to A_{\mathcal{C}}(n)/\langle u_{ij} \mid i \neq j\rangle$. Then there is a commuting diagram

$$\begin{array}{ccc} (A_o(n), u_n^o) & \longrightarrow & \mathrm{C}^*_{\max}(\mathbb{Z}_2^{*n}) \\ \downarrow & & \downarrow \\ (A_{\mathcal{C}}(n), u_n) & \xrightarrow{\pi} & \mathrm{C}^*_{\max}(H)\,. \end{array}$$

So H is the universal group generated by elements $a_i = \pi(u_{ii})$, $1 \leqslant i \leqslant n$ of order two that satisfy $T_p \otimes 1 = (\pi(u_n))^{\otimes k}(T_p \otimes 1)$ for all $p \in \mathcal{C}$. Moreover, it suffices to consider partitions $p \in \mathcal{C}$ on one row. Let $p \in \mathcal{C}$ be such a partition of length k. Then $T_p \otimes 1 = u^{\otimes k}(T_p \otimes 1)$. Hence, we have equality of

$$(T_p \otimes 1)(1 \otimes 1) = \sum_{i(1),\ldots,i(k)=1}^{n} \delta_p(i) \cdot e_{i(1)} \otimes \cdots \otimes e_{i(k)} \otimes 1$$

and

$$u^{\otimes k}(T_p \otimes 1)(1 \otimes 1) = \sum_{\substack{i(1),\ldots,i(k),\\ j(1),\ldots,j(k)=1}}^{n} \delta_p(j) \cdot e_{i(1)} \otimes \cdots \otimes e_{i(k)} \otimes u_{i(1)j(1)} \cdots u_{i(k)j(k)}\,.$$

Applying the projection π to both equations, we obtain that for all $i(1), \dots, i(k)$ with $\delta_p(i) = 1$ we have $\pi(u_{i(1)i(1)}) \cdots \pi(u_{i(k)i(k)}) = 1$. This shows that $H \cong \mathbb{Z}^{*n}/F(\mathcal{C})_n$. $\square$

Note that as a consequence of the last theorem the C*-algebra $A_{\mathcal{C}}(n)$ is a canonical extension of the full group C*-algebra $C^*_{\max}(\mathbb{Z}_2^{*n}/F(\mathcal{C})_n)$.

Lemma 6.6.4. *Let H be a subgroup of $\mathbb{Z}_2^{*\infty}$. Then $\mathbb{Z}_2^{*\infty}/H \cong \varinjlim(\mathbb{Z}_2^{*n}/(H)_n, \phi_n)$, where ϕ_n is defined by the diagram*

$$\begin{array}{ccc} (H)_n & \xrightarrow{\phi_n} & (H)_{n+1} \\ \cap & & \cap \\ \mathbb{Z}_2^{*n} & \xrightarrow{a_i \mapsto a_i} & \mathbb{Z}_2^{*n+1} . \end{array}$$

Proof. By universality of inductive limits, we have to show that for any compatible family of morphisms

$$\begin{array}{ccccccc} \mathbb{Z}_2^{*n}/(H)_n & \longrightarrow & \mathbb{Z}_2^{*n+1}/(H)_{n+1} & \longrightarrow & \mathbb{Z}_2^{*n+2}/(H)_{n+2} & \longrightarrow & \cdots \\ & \searrow & & \searrow & & \searrow & \\ & & & & & & K \end{array}$$

the induced map $\pi : \mathbb{Z}_2^{*\infty} \to K$ contains H in its kernel. This follows from the fact that $H = \bigcup_{n \geqslant 1} (H)_n$. $\square$

The following corollary says that for a simplifiable category of partitions, we can recover $F(\mathcal{C})$ directly from the diagonal subgroups of the family $(A_{\mathcal{C}}(n), u_n)_{n \geqslant 2}$.

Corollary 6.6.5. *Let $\mathcal{C}$ be a simplifiable category of partitions. Then $F(\mathcal{C}) = \ker(\mathbb{Z}_2^{*\infty} \to \varinjlim(\mathrm{Diag}(A_{\mathcal{C}}(n), u_n)))$.*

Proof. This follows from Theorem 6.6.3 and Lemma 6.6.4. $\square$

We can also recover $(A_{\mathcal{C}}(n), u_n)$ directly from the group $F(\mathcal{C})$ without passing through the framework of partitions. This is done by considering quantum isometry groups. By [17] the category $\langle \sqcup\!\!\!/\sqcap \rangle$ gives rise to an easy quantum group, denoted by $H_n^{[\infty]}$. It is the maximal simplifiable quantum group.

Theorem 6.6.6. *Let $H \leqslant E \leqslant \mathbb{Z}_2^{*\infty}$ be a proper S_0-invariant subgroup of E. Then the maximal quantum subgroup of $(\mathcal{C}(H_n^{[\infty]}), u_{\mathrm{simpl}})$ acting faithfully and isometrically on $C^*_{\max}(\mathbb{Z}_2^{*n}/(H)_n)$ is $(A_{\mathcal{C}_H}(n), u_n)$.*

Proof. Denote by $\pi : \mathbb{Z}_2^{*n} \to \mathbb{Z}_2^{*n}/(H)_n$ the canonical quotient map and by $a_1, \dots, a_n$ the canonical generators of $\mathbb{Z}_2^{*n}$. Since $H_n^{[\infty]}$ is isomorphic to its coopposite quantum group by Corollary 6.5.6, we may consider quantum subgroups of $(H_n^{[\infty]})^{\mathrm{coop}}$ instead of $H_n^{[\infty]}$ in the following. Let (A, u) be a compact quantum subgroup of $H_n^{[\infty]}$ such that (A, u^t) acts faithfully by

$$\alpha : \mathrm{C}^*_{\max}(\mathbb{Z}_2^{*n}/(H)_n) \to \mathrm{C}^*_{\max}(\mathbb{Z}_2^{*n}/(H)_n) \otimes A : \pi(a_i) \to \sum_j \pi(a_j) \otimes u_{ij}$$

on $\mathrm{C}^*_{\max}(\mathbb{Z}_2^{*n}/(H)_n)$ and preserves the length function l associated with the generators $\pi(a_1), \dots, \pi(a_n)$. We show that (A, u) is a quantum subgroup of $(A_{\mathcal{C}_H}(n), u_n)$.

Since (A, u) is a quotient of $(C(H_n^{[\infty]}), u_{\mathrm{simpl}})$, Proposition 6.5.2 shows that the entries u_{ij} of u are self-adjoint partial isometries, which are pairwise orthogonal in every row and in every column and whose support projections are central. We check the additional relations that are imposed on the u_{ij} by the fact that (A, u^t) acts isometrically on $\mathrm{C}^*_{\max}(\mathbb{Z}_2^{*n}/(H)_n)$. For every word $a_{i(1)} \cdots a_{i(k)} \in \mathbb{Z}_2^{*n}$ we have

$$\alpha(\pi(a_{i(1)} \cdots a_{i(k)})) = \sum_{j(1), \dots, j(k)=1}^{n} \pi(a_{j(1)} \cdots a_{j(k)}) \otimes u_{i(1)j(1)} \cdots u_{i(k)j(k)} \,.$$

This expression has non-zero coefficients $u_{i(1)j(1)} \cdots u_{i(k)j(k)}$ only for those $(j(1), \dots, j(k))$ with $l(\pi(a_{j(1)} \cdots a_{j(k)})) = l(\pi(a_{i(1)} \cdots a_{i(k)}))$. Using the fact that $l(\pi(a_{i(1)} \cdots a_{i(k)})) = 0$ if and only if $a_{i(1)} \cdots a_{i(k)} \in (H)_n$, this means in particular that for all $a_{i(1)} \cdots a_{i(k)} \notin (H)_n$ we have

$$\sum_{a_{j(1)} \cdots a_{j(k)} \in (H)_n} u_{i(1)j(1)} \cdots u_{i(k)j(k)} = 0 \,.$$

On the other hand, if $a_{i(1)} \cdots a_{i(k)} \in (H)_n$ then

$$\alpha(\pi(a_{i(1)} \cdots a_{i(k)})) = \alpha(1) = 1 \otimes 1 \,.$$

It follows that

$$\sum_{a_{j(1)} \cdots a_{j(k)} \in (H)_n} u_{i(1)j(1)} \cdots u_{i(k)j(k)} = 1 \,.$$

We proved that for all $a_{i(1)} \cdots a_{i(k)} \in \mathbb{Z}_2^{*n}$ we have

$$\sum_{a_{j(1)} \cdots a_{j(k)} \in (H)_n} u_{i(1)j(1)} \cdots u_{i(k)j(k)} = \delta_p(i) \cdot 1\,,$$

where p denotes the partition in $\mathcal{C}_H$ that is associated with $a_{i(1)} \cdots a_{i(k)}$.

Next, we claim that if $u_{i(1)j(1)} \cdots u_{i(k)j(k)} \neq 0$, then the word $a_{j(1)} \cdots a_{j(k)}$ arises from $a_{i(1)} \cdots a_{i(k)}$ by a permutation of letters. Recall that all u_{ij} are self-adjoint partial isometries, which are pairwise orthogonal in every row and every column and whose supports are central in $A_{\mathcal{C}_H}(n)$. If $1 \leqslant \alpha, \beta \leqslant k$, we can conclude from $i(\alpha) = i(\beta)$ and $u_{i(1)j(1)} \cdots u_{i(k)j(k)} \neq 0$, that $j(\alpha) = j(\beta)$. Similarly, we conclude from $j(\alpha) = j(\beta)$, that $i(\alpha) = i(\beta)$. This proves our claim.

Take now a partition $p \in \mathcal{C}$. By our claim, for any multi-index $i = (i(1), \ldots, i(k))$, which satisfies $\delta_p(i) = 0$, we have

$$\sum_{j(1),\ldots,j(k)=1}^{n} \delta_p(j) \cdot u_{i(1)j(1)} \cdots u_{i(k)j(k)} = 0$$

If $i = (i(1), \ldots, i(k))$ satisfies $\delta_p(i) = 1$, we obtain, using the claim again,

$$\sum_{j(1),\ldots,j(k)=1}^{n} \delta_p(j) \cdot u_{i(1)j(1)} \cdots u_{i(k)j(k)}$$

$$= \sum_{a_{j(1)} \cdots a_{j(k)} \in (H)_n} u_{i(1)j(1)} \cdots u_{i(k)j(k)} = 1\,.$$

Summarising, we have

$$\sum_{j(1),\ldots,j(k)=1}^{n} \delta_p(j) \cdot u_{i(1)j(1)} \cdots u_{i(k)j(k)} = \delta_p(i)\,.$$

We infer that

$$u^{\otimes k}(T_p \otimes 1)(1 \otimes 1) = \sum_{\substack{i(1),\ldots,i(k),\\ j(1),\ldots,j(k)=1}}^{n} e_{i(1)} \otimes \cdots \otimes e_{i(k)} \otimes \delta_p(j) \cdot u_{i(1)j(1)} \cdots u_{i(k)j(k)}$$

$$= \sum_{i(1),\ldots,i(k)=1}^{n} e_{i(1)} \otimes \cdots \otimes e_{i(k)} \otimes \left(\sum_{j(1),\ldots,j(k)=1}^{n} \delta_p(j) \cdot u_{i(1)j(1)} \cdots u_{i(k)j(k)} \right)$$

$$= \sum_{i(1),\ldots,i(k)=1}^{n} e_{i(1)} \otimes \cdots \otimes e_{i(k)} \otimes \delta_p(i) \cdot 1$$

$$= (T_p \otimes 1)(1 \otimes 1)\,.$$

We proved that (A, u) is a quantum subgroup of $(A_{\mathcal{C}_H}(n), u_n)$.

Next we show that $(A_{\mathcal{C}_H}(n), (u_n)^t)$ acts faithfully and isometrically on $\mathrm{C}^*_{\max}(\mathbb{Z}_2^{*n}/(H)_n)$. The map

$$\alpha : \mathrm{C}^*_{\max}(\mathbb{Z}_2^{*n}/(H)_n) \to \mathrm{C}^*_{\max}(\mathbb{Z}_2^{*n}/(H)_n) \otimes A_{\mathcal{C}_H}(n) : \pi(a_i) \mapsto \sum_j \pi(a_j) \otimes u_{ij}$$

is a well defined action of $(A_{\mathcal{C}_H}(n), (u_n)^t)$ on $\mathrm{C}^*_{\max}(\mathbb{Z}_2^{n}/(H)_n)$, by the calculations in the first part of the proof. By definition, it is faithful, so it remains to show that it is isometric. We say that a word $a_{i(1)} \cdots a_{i(k)}$ is reduced in $\mathbb{Z}_2^{*n}/(H)_n$, if there is no $k' < k$ and $j(1), \ldots, j(k')$ such that $\pi(a_{i(1)} \cdots a_{i(k)}) = \pi(a_{j(1)} \cdots a_{j(k')})$. A word $a_{i(1)} \cdots a_{i(k)}$ is reduced in $\mathbb{Z}_2^{*n}/(H)_n$ if and only if $l(\pi(a_{i(1)} \cdots a_{i(k)})) = k$. Denoting

$$L_k = \mathrm{span}\{w = \pi(a_{i(1)} \cdots a_{i(k)}) \,|\, a_{i(1)} \cdots a_{i(k)} \text{ is reduced as a word in } \mathbb{Z}_2^{*n}/(H)_n\}\,,$$

we have to show that $\alpha(L_k) \subset L_k \otimes A_{\mathcal{C}_H}(n)$. First note that $\alpha(L_k) \subset \bigcup_{k' \leqslant k} L_{k'} \otimes A_{\mathcal{C}_H}(n)$. Let $\pi(a_{i(1)}) \cdots \pi(a_{i(k)})$ denote a word that is reduced in $\mathbb{Z}_2^{*n}/(H)_n$, write $w = \pi(a_{i(1)} \cdots a_{i(k)})$ and assume that $\alpha(w) \notin L_k \otimes A_{\mathcal{C}_H}(n)$. We have

$$\alpha(w) = \sum_{j(1),\ldots,j(k)} \pi(a_{j(1)} \cdots a_{j(k)}) \otimes u_{i(1)j(1)} \cdots u_{i(k)j(k)}\,,$$

so there is some mutiindex $(j(1), \ldots, j(k))$ such that

- $a_{j(1)} \cdots a_{j(k)}$ is not reduced in $\mathbb{Z}_2^{*n}/(H)_n$ and
- $u_{i(1)j(1)} \cdots u_{i(k)j(k)} \neq 0$.

As seen above, the word $a_{i(1)} \cdots a_{i(k)}$ must arise as a permutation of letters from $a_{j(1)} \cdots a_{j(k)}$, because $u_{i(1)j(1)} \cdots u_{i(k)j(k)} \neq 0$. Since $a_{j(1)} \cdots a_{j(k)}$ is not reduced in $\mathbb{Z}_2^{*n}/(H)_n$ and $a_{i(1)} \cdots a_{i(k)}$ arises from $a_{j(1)} \cdots a_{j(k)}$ by a permutation of letters, also $a_{i(1)} \cdots a_{i(k)}$ is not reduced in $\mathbb{Z}_2^{*n}/(H)_n$. This is a contradiction. We proved that α is an isometric action of $(A_{\mathcal{C}_H}(n), (u_n)^t)$. Summarising we showed that the maximal quantum subgroup of $(\mathrm{C}(H_n^{[\infty]}), u_{\mathrm{simpl}})$ which acts faithfully and isometrically on $\mathrm{C}^*_{\max}(\mathbb{Z}_2^{*n}/(H)_n)$ is isomorphic to $(A_{\mathcal{C}_H}(n), (u_n)^t)$. Invoking Corollary 6.5.6, we see that $(A_{\mathcal{C}_H}(n), (u_n)^t) \cong (A_{\mathcal{C}_H}(n), u_n)$ and this finishes the proof. □

In view of the last theorem, it would be interesting to calculate the full quantum isometry groups of $\mathrm{C}^*_{\max}(\mathbb{Z}^{*n}/(H)_n)$.

Example 6.6.7. One class of groups which appear as $\mathbb{Z}_2^{*n}/(H)_n$ for some S_0-invariant subgroup $H \leqslant E \leqslant \mathbb{Z}_2^{*\infty}$ are Coxeter groups. A Coxeter group G is of the above form if and only if

$$G = \mathbb{Z}_2^{*n}/\langle (a_i a_j)^s \mid 1 \leqslant i, j \leqslant n \rangle ,$$

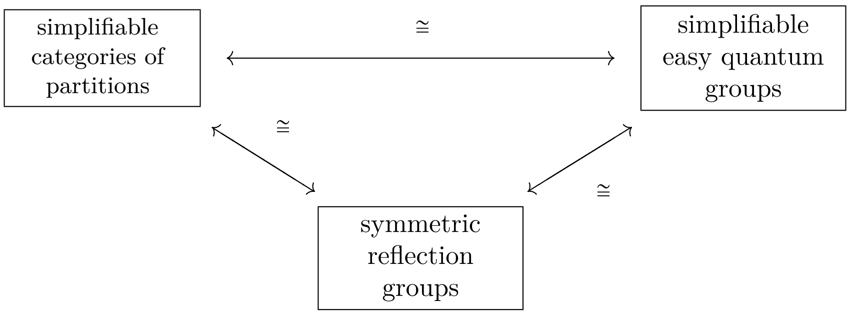
for some $s \in \mathbb{N}_{\geqslant 2}$. The easy quantum group associated with it is $H_n^{[s]}$, since the category of partitions of the latter is given by $\langle ⊓⊓, h_s \rangle$.

6.6.1 The triangluar relationship between quantum groups, reflection groups and categories of partitions

Let us give a name to the groups that appear in the first part of this section.

Definition 6.6.8. Let $H \leqslant \mathbb{Z}_2^{*\infty}$ be an S_0-invariant subgroup. A *symmetric reflection group* G is the quotient of $\mathbb{Z}_2^{*n}$ by the intersection $H \cap \mathbb{Z}_2^{*n}$. The images of the canonical generators of $\mathbb{Z}_2^{*n}$ in G are called the generators of the symmetric reflection group G.

We obtain a correspondence between simplifiable quantum groups and symmetric reflection groups. By the work of Banica and Speicher [24], there is a correspondence between easy quantum groups and categories of partitions. Finally, the results of Section 6.3 show that there is a correspondence between symmetric reflection groups and simplifiable categories of partitions. We therefore obtain a triangular correspondence between simplifiable quantum groups, symmetric reflection groups and simplifiable categories of partitions that we are going to recap.



From quantum groups to categories of partitions and back: By the work of Banica and Speicher [24] there is a one-to-one correspondence between easy quantum groups and categories of partitions. As described in Section 6.1.2, the category of partitions associated with an easy quantum group (A, u) describes the intertwiner spaces between tensor powers of the fundamental corepresentation u. By definition, an easy quantum groups is called a simplifiable quantum group, if the category of partitions associated with it contains the four block ⊓⊓⊓, the pair positioner partition ⊔⁄⊓, but not the double singleton $\uparrow \otimes \uparrow$.

From categories of partitions to reflection groups and back: Theorem 6.3.10 shows that there is a one-to-one correspondence between categories of partitions and S_0-invariant subgroups of $\mathbb{Z}_2^{*\infty}$. By definition, symmetric reflection groups on infinitely many generators correspond precisely to the S_0-invariant subgroups of $\mathbb{Z}_2^{*\infty}$.

From quantum groups to reflection groups and back: Let (A, u), $u \in \mathrm{M}_n(\mathbb{C})(A)$ be a compact matrix quantum group. The quotient of A by the ideal generated by $\{u_{ij} \mid i \neq j\}$ is a cocommutative compact matrix quantum group. It is of the form $(\mathrm{C}^*(G), (\delta_{ij} g_i)_{ij})$ for a generating set $g_1, \ldots, g_n$ of a discrete group G. Theorem 6.6.3 shows that if $(\mathrm{C}(H), u)$ is a simplifiable quantum group, then the associated discrete group G with generators $g_1, \ldots, g_n$ is a symmetric reflection group. Vice versa, Theorem 6.6.6 associates with a symmetric reflection group G on finitely many generators $g_1, \ldots, g_n$ the maximal quantum subgroup of $H \subset H_n^{[\infty]}$ that acts faithfully and isometrically on $\mathrm{C}^*_{\max}(G)$. The remark after Theorem 6.6.3 says that $\mathrm{C}(H)$ is a canonical extension of $\mathrm{C}^*_{\max}(G)$ as a C^*-algebra.

From finitely generated to infinitely generated symmetric reflection

groups and back: Given a symmetric reflection group G on infinitely many generators $g_1, g_2, \ldots$ all groups $G_n = \langle g_1, g_2, \ldots g_n \rangle$ are symmetric reflection groups also. They form an inductive system

$$\cdots \hookrightarrow G_n \hookrightarrow G_{n+1} \hookrightarrow \cdots$$

of symmetric reflection groups. We can obtain G as the inductive limit of this system.

Theorems 6.6.3 and 6.6.6 show that the above correspondences are compatible with each other. Put differently, the triangle between quantum groups, discrete groups and categories of partitions commutes.

Chapter 7

Summary and open problems

In this chapter, we give a brief summary of the work presented in this thesis in Section 7.1 and then in Section 7.2 sketch open problems further supporting the common direction of our projects.

7.1 Summary of our results

This thesis collects different results on von Neumann algebras and quantum groups focusing on classification results for natural subclasses of these and calculation of categories of representations. The work in Chapter 3 gives a complete classification of group measure space constructions associated with Bernoulli actions and some of their quotients of free groups of finite rank in terms of the rank of the free groups involved. Chapters 5 and 6 contain classification results for von Neumann algebras and easy quantum groups in terms of classical data. While in Chapter 5 partial classifiaction results for free Bogoliubov crossed products in terms of an orthogonal representation of $\mathbb{Z}$, from which the crossed product was constructed, is obtain, in Chapter 6 the main point is the identification of a completely new invariant, namely symmetric reflection groups, parametrising simplifiable easy quantum groups. In the latter chapter, we put the lattice structure of the set of all easy quantum groups in the focus and show that it is not possible to describe it completely, by injecting the lattice of varieties of groups. This point of view also allows us to make non-trivial statements on the structure of easy quantum groups, demonstrating the direct connection between classification and structural results.

Categories of representations are investigated in Chapters 2 and 4. In Chapter 2, the fusion rules of certain free quantum groups were calculated. Such calculations, together with other methods already known before, hopefully allow us to find the fusion rules of certain subclasses of the simplifiable easy quantum groups described before. In Chapter 4, the existence of II_1 factors with prescribed bimodule categories, for example arbitrary finite tensor C^*-categories, is shown. The tensor C^*-category we can realise as a bimodule category can also be the category of unitary finite dimensional corepresentations of a discrete quantum group from a fairly big class. This connects our work on bimodule categories of II_1 factors with our work on quantum groups, as we will describe more detailed in Section 7.2.2. Our results on bimodule categories allow us to obtain calculations of other invariants for II_1 factors, proving the usefulness of this approach.

7.2 Open problems

We sketch several open problems deepening our research and relating the different topics treated in this thesis.

7.2.1 Fusion rules for easy quantum groups

In view of older success when calculating fusion rules on a combinatorial basis [7, 25] as in Chapter 2, it is realistic to expect that fusion rules of any easy quantum group are at least in principle calculable. The results of Chapter 6, however, show that the class of easy quantum groups is very rich, rendering a uniform approach to all easy quantum groups quite impossible. It is therefore reasonable to select interesting subclasses of easy quantum groups and try to calculate their fusion rules. The higher hyperoctahedral and the higher hyperoctahedral series introduced in [17] are a good candidates for such classes, since they are very natural from a combinatorial point of view and correspond under the bijection described in Chapter 6 at the same time to the variety of abelian groups with fixed exponent and the variety of all groups with fixed exponent, respectively. The associated reflection groups are Coxeter groups, so that methods for determining the size of balls in their Cayley graph are available. Since the size of such balls is related to dimensions of intertwiner spaces of the corresponding easy quantum group, it is probable that the fusion rules of the higher hyperoctahedral quantum groups can be calculated combining group theoretical results and the known combinatorial methods.

7.2.2 New examples of bimodule categories of II_1 factors

As shown in Chapter 4, every faithful corepresentation of a discrete Kac algebra in the hyperfinite II_1 factor enables us to realise the category of its finite dimensional unitary corepresentations as the bimodule category of a II_1 factor. In view of the problem described in Section 7.2.1, it is interesting to find a faithful corepresentation of the discrete dual of $\mathrm{A_o}(n)$ in the hyperfinite II_1 factor. In fact, it reasonable to conjecture that $\widehat{\mathrm{A_o}(n)}$ is maximally almost periodic in the sense of Sołtan [198], due to its similarity with free groups. It would be useful to obtain a proof that is based on the combinatorics of intertwiner spaces of $\mathrm{A_o}(n)$, so that also the following question can be answered: are all discrete duals of easy quantum groups maximally almost periodic? A positive answer to this question would yield many new examples of bimodule categories combining the work in Chapters 4 and 6.

7.2.3 Calculation of bimodule categories of II_1 factors with non-trivial fundamental group

In Chapter 4 we could show that many compact tensor C*-categories arise as the bimodule category of a II_1 factor. However, all factors that appear there have a trivial fundamental group. Skandalis asked whether it would be possible to obtain examples with known bimodule category and non-trivial fundamental group. As explained in Section 1.1.6, the group of invertible objects of the bimodule category of a II_1 factor M is related to its fundamental group by the exact sequence

$$1 \to \mathrm{Out}(M) \to \mathrm{Grp}(M) \to \mathcal{F}(M) \to 1\,.$$

In this light, it would be interesting to find for any countable subgroup $S \leqslant \mathbb{R}_{>0}$ and every finite tensor C*-category $\mathcal{C}$ a II_1 factor M such that $\mathcal{F}(M) = S$ and $\mathrm{Bimod}(M) \cong \mathcal{S} \boxtimes \mathcal{C}$, where $\mathcal{S}$ is the category of finite dimensional unitary representations of $\widehat{S}$ and $\boxtimes$ denotes the Deligne tensor product of tensor categories [61].

7.2.4 Is every countable tensor C*-category the bimodule category of a II_1 factor?

A compact tensor C*-category is called countable, if it has a countable number of isomorphism classes of irreducible objects. In view of our result in Chapter 4 that every finite tensor C*-category is the bimodule category of a II_1 factor, it is natural to ask whether the same is true for countable tensor C*-categories. Note also that by a result of Vaes every countable group is the outer automorphism group of a II_1 factor [215].

Recently, in [43], it was shown that every countable tensor C*-category can be embedded into the category of bimodules over a free group factor. It seems plausible, that every countable tensor C*-category can be even realised as the category of bimodules of a depth 2 subfactor inclusion of a free group factor into a factor of type II_∞. Now the methods Chapter 4 could possibly apply, if the following theorem of Popa on the outer automorphism group of the hyperfinite II_1 factor, could be generalised to free group factors. In [117] it is shown that for all countable subgroups $G, H \subset \mathrm{Out}(R)$, there is an element $\alpha \in \mathrm{Out}(R)$ such that G and $(\mathrm{Ad}\,\alpha)(H)$ are free. See [216] for the appropriate generalisation to bimodule categories. This problem motivates questions about the outer automorphism group of free group factors, one of which we state in 7.2.5.

7.2.5 Strong solidity of crossed products of integer actions on free group factors

In Chapter 5, we extended the results of Houdayer-Shylakhtenko strong solidity of free Bogoliubov crossed products of the integers. We showed that free Bogoliubov crossed products of a direct sum of some mixing representation with an at most one-dimensional representation is strongly solid, while free Bogoliubov crossed products of actions with two-dimensional rigid subspaces are not even solid. It would be interesting to explore, whether a crossed product of a free group factor by an integer action is generically strongly solid.

There are classes of probability measure transformations in which a generic (in the sense of descriptive set theory [127]) action is weakly mixing and rigid [1]. If $\mathbb{Z} \overset{\alpha}{\curvearrowright} X$ is such a transformation, then $\mathbb{Z} \to \mathcal{O}(\mathrm{L}^2{}_{\mathbb{R}}(X) \ominus \mathbb{R} \cdot 1)$ is a weakly mixing and rigid representation of $\mathbb{Z}$ and as such it gives rise to a free Bogoliubov crossed product with property Γ, which hence cannot be solid. On the other hand, it is known that a generic (in the sense of random walks) automorphism of a finite rank free group is iwip (irreducible with irreducible powers) [185, 197] and that mapping cones $\mathbb{F}_n \rtimes \mathbb{Z}$ of iwip automorphisms of a free group are hyperbolic [29, 28]. Combining this with the fact that group von Neumann algebras of hyperbolic groups are strongly solid [46], we found a class of automorphisms of free group factors, which generically give rise to strongly solid crossed products.

As mentioned earlier in Section 7.2.4, specifically in the context of a possible generalisation of the results in Chapter 4, it is interesting to investigate the outer automorphism group of free group factors from as many perspectives as possible. The problem just described could be one aspect of such investigations.

7.2.6 Actions of duals of free orthogonal groups

The description of this problem is highly speculative. As explained in Section 1.2, actions of quantum groups provide a link between operator algebras and quantum groups. One interesting problem in this context, is the search for actions of quantum groups on abelian von Neumann algebras. Indeed, there are no natural sources of such actions of quantum groups. However, in the more specific situation of duals of free orthogonal quantum groups, due to their universality properties, one can hope to find examples. Assume that there was a strictly outer ergodic trace preserving action of $\widehat{O_n^+}$ on some diffuse abelian von Neumann algebra A. Then $A \subset A \rtimes \widehat{O_n^+}$ is a maximally abelian, quasi-regular subalgebra of a II_1 factor. By a result of Popa [166, Lemma 3.5], A is a Cartan subalgebra. So the construction of Feldman and Moore [89, 90] associates an ergodic II_1 equivalence relation with $A \subset A \rtimes \widehat{O_n^+}$.

Several question about this equivalence relation would arise, if one could construct strictly outer ergodic trance preserving actions of $\widehat{O_n^+}$ on abelian von Neumann algebras. Can two equivalence relations arising this way from $\widehat{O_n^+}$ and $\widehat{O_m^+}$ ever be isomorphic, if $n \neq m$? This question is of course inspired by the fact that free ergodic pmp actions of free groups of different rank can never be isomorphic. The von Neumann algebras $\mathrm{L}^\infty(O_n^+)$ of free orthogonal quantum groups posses many properties of free group factors. They are for example strongly solid [119] and posses the W*-completely contractive approximation property [42, 91]. At the moment, it seems not possible to decide whether $\mathrm{L}^\infty(O_n^+)$ are free group factors or not. However, in the previous setting, it would be interesting to look for actions of $\widehat{O_n^+}$ yielding an equivalence relation that is stably isomorphic to an orbit equivalence relation of a free ergodic pmp action of a free group. Such a finding could be interpreted as a measure equivalence result between free groups and duals of free orthogonal quantum groups.

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Curriculum vitae

Sven Raum was born in Kassel, Germany, on 9th April 1985. He visited the Grotefend Gymnasium Münden and obtained his general qualification for university entrance in 2004. From October 2004 until April 2005 he served in the German military and did civilian service. In October 2005 he started his studies in mathematics with a minor in business administration at the Westfälische Wilhelmsunität Münster, Germany. He obtained his Diplom with honours in May 2009. In September 2009 he started to follow the PhD programme of the Katholieke Universiteit Leuven, Beglium under the supervision of Stefaan Vaes. From 2007 until 2013 Sven Raum taught several courses on mathematical topics for mathematicians and non-mathematicians. From 2009 until 2012 he co-supervised three master theses together with Stefaan Vaes. He speaks German, English, French and Dutch.

List of publications

Papers

- Tensor C^*-categories arising as bimodule categories of II_1 factors, with Sébastien Falguières, Adv. in Math. 237, (2013), 331-359.
- Stable orbit equivalence of Bernoulli actions of free groups and isomorphism of some of their factor actions, with Niels Meesschaert and Stefaan Vaes, to appear in Expositiones Mathematica.
- Isomorphisms and Fusion Rules of Orthogonal Free Quantum Groups and their Complexifications, Proc. Amer. Math. Soc. 140 (2012), no. 9, 3207-3218.

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- Amalgamated free product type III factors with at most one Cartan subalgebra, with Rémi Boutonnet and Cyril Houdayer, arXiv:1212.4994, 2012.
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