Exercises on Homology and Cohomology

Spring term 2018, Sheet 1

Hand in before 10 o'clock on 26th February 2018
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Exercise 1

In this exercise we investigate the independence of the Euler characteristic from concrete Δ-complex structures.

(i) Let X be a topological space admitting a finite 2-dimensional Δ-complex structure. Show that the Euler characteristic does not depend on the choice of the particular Δ-complex structure.

(ii) Define a notion of Euler characteristic for topological spaces with arbitrary finite Δ-complexes and show that it does not depend on the particular choice of Δ-complex structure.

Exercise 2 (will be corrected)

In this exercise we calculate the Euler characteristic of topological surfaces, which generalise the torus to objects with "several wholes":

We write Σ_g for the standard surface of genus $g \in \mathbb{N}$, as defined subsequently. The standard surface of genus 0 is the sphere. For $g \in \mathbb{N}_{\geq 1}$, the standard surface of genus g is the quotient of a regular $4g$-polygon whose edges are identified according to the following rule: $a_1, b_1, a_1^{-1}, b_1^{-1}, a_2, b_2, a_2^{-1}, b_2^{-1}, \ldots, a_g, b_g, a_g^{-1}, b_g^{-1}$. Here a_i designates an edge with label a_i oriented in the mathematically positive sense and a_i^{-1} designates an edge with label a_i oriented in the mathematically negative sense.

(i) Find a finite 2-dimensional Δ-complex structure on each standard surface.

(ii) Calculate the Euler characteristic $\chi(\Sigma_g)$ of the standard surface of genus g as a function of g.

Exercise 3 (will be corrected)

In this exercise we see the differences between the notions of injectivity/monomorphism and surjectivity/epimorphism by means of examples.

(i) Show that in the category $\mathbf{Ring}$ of rings, the inclusion $\mathbb{Z} \subset \mathbb{Q}$ defines an epimorphism.

(ii) Show that in the category $\mathbf{Top}_{\mathrm{Haus}}$ of Hausdorff topological spaces, the inclusion $\mathbb{Q} \subset \mathbb{R}$ defines an epimorphism.

(iii) Show that in the category $\mathbf{Div}$ of divisible groups the quotient map $\mathbb{Q} \to \mathbb{Q}/\mathbb{Z}$ defines a monomorphism. (A divisible group is an abelian group A such that for every $a \in A$ and every $n \in \mathbb{N}_{\geq 1}$ there is some $b \in A$ satisfying $nb = a$.)

Exercise 4

Let k be a field and denote by $\mathrm{Free} : \mathbf{Set} \to k\text{-}\mathbf{Vec}$ the functor associating with a set $X \in \mathbf{Set}$ the free vector space k^X. Show that Free is essentially surjective, that is for every $V \in k\text{-}\mathbf{Vec}$ there is $X \in \mathbf{Set}$ such that $V \cong \mathrm{Free}(X)$.