Exercises on Homology and Cohomology

Spring term 2018, Sheet 11

Hand in before 10 o'clock on 14th May 2018
Mailbox of Sven Raum in MA B2 475

Sven Raum
Haoqing Wu

Exercise 1 (easy)
Let G be a group and denote by $I \trianglelefteq \mathbb{Z}G$ the kernel of $\epsilon : \mathbb{Z}G \to \mathbb{Z}$.

(i) Show that the elements $(1 - u_g)_{g \in G \setminus \{e\}}$ form a basis of I as a $\mathbb{Z}$-module.

(ii) If S is set of generators of G, show that the elements $\{1 - u_s\}_{s \in S}$ generate I as a $\mathbb{Z}G$-module.

(iii) Conclude that if G is finitely generated, then $\mathbb{Z}$ admits a free resolution $F_* \to \mathbb{Z}$ for which F_1 has finite rank.

Exercise 2 (medium)
Let $G = \mathbb{Z}^n$ and consider the action $G \curvearrowright \mathbb{R}^n$ by translation.

(i) Find a suitable structure of a CW-complex such that $\mathbb{R}^n$ becomes a G-CW-complex.

(ii) Construct a free resolution F_* of the trivial $\mathbb{Z}G$-module $\mathbb{Z}$ of length n.

(iii) Calculate $\mathrm{H}^n(G) = \mathrm{H}^n(\mathrm{Hom}(F_*, \mathbb{Z}))$.

Exercise 3 (difficult)
Let G be a group and denote by $I \trianglelefteq \mathbb{Z}G$ the kernel of $\epsilon : \mathbb{Z}G \to \mathbb{Z}$.

(i) Show that if a family $(1 - u_s)_{s \in S}$ generates I as a $\mathbb{Z}G$-module, then S is a generating set for G.

(ii) Show that if I is finitely generated as a $\mathbb{Z}G$-module, then G is finitely generated.

(iii) Show that the trivial $\mathbb{Z}G$-module $\mathbb{Z}$ admits a free resolution $F_* \to \mathbb{Z}$ with F_1 of finite rank if and only if G is finitely generated.