Exercises on Homology and Cohomology

Spring term 2018, Sheet 12

Hand in before 10 o'clock on 28th May 2018
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Exercise 1 (easy)
In this exercise we characterise which groups have a projective resolution of length 0, that is $P_* \to \mathbb{Z}$ such that $P_n = 0$ for all $n \geq 1$. More precisely, let G be a group and $\epsilon : \mathbb{Z}G \to \mathbb{Z}$ the augmentation map, which satisfies $\epsilon(u_g) = 1$ for all $g \in G$.

(i) Show that ϵ splits (i.e. there is a $\mathbb{Z}G$-modular map $\varphi : \mathbb{Z} \to \mathbb{Z}G$ such that $\epsilon \circ \varphi = \mathrm{id}_{\mathbb{Z}}$) if and only if G is trivial.

(ii) Characterise those groups G for which $\mathbb{Z}$ is a projective $\mathbb{Z}G$-module.

(iii) Characterise those groups G for which there is a projective resolution of length 0 of $\mathbb{Z}$.

Exercise 2 (medium)
Let G be a group and $F_* \to \mathbb{Z}$ be the bar resolution of the trivial module $\mathbb{Z}$ over $\mathbb{Z}G$ and recall that a $\mathbb{Z}G$-basis of F_n is given by the family $[g_1|\cdots|g_n]$ for $g_1, \ldots, g_n \in G$. Let $\overline{F_n}$ be the quotient by of F_n by the free $\mathbb{Z}G$-submodule whose basis is $\{[g_1|\cdots|g_n] \mid \exists i \in \{1, \ldots, n\} : g_i = e\}$.

(i) Show that the boundary maps of F_* induce well-defined boundary maps for $\overline{F_*}$.

(ii) Show that $\overline{F_*} \to \mathbb{Z}$ is a free resolution over $\mathbb{Z}G$. It is called the normalised bar resolution of $\mathbb{Z}$ over $\mathbb{Z}G$.

Exercise 3 (medium)
Show that the following statements are equivalent for an R-module A.

(i) A is a projective R-module.

(ii) $\mathrm{Hom}_R(A, -)$ is an exact functor, that is for every short exact sequence of R-modules

$$0 \to M_1 \to M_2 \to M_3 \to 0$$

also the sequence of abelian groups

$$0 \to \mathrm{Hom}_R(A, M_1) \to \mathrm{Hom}_R(A, M_2) \to \mathrm{Hom}_R(A, M_3) \to 0$$

is exact.