Exercises on Homology and Cohomology

Spring term 2018, Sheet 2

Hand in before 10 o'clock on 5th March 2018
Mailbox of Sven Raum in MA B2 475

Sven Raum
Haoqing Wu

Exercise 1

In this exercise we investigate different notions of triviality of a chain complex. Let (C, d) be a chain complex of R-modules. Show that the following statements are equivalent.

(i) C is exact, that is $\ker d_{n-1} = \operatorname{im} d_n$ for all $n \in \mathbb{Z}$.

(ii) C is acyclic, that is $\mathrm{H}_n(C) = 0$ for all $n \in \mathbb{Z}$.

(iii) The unique chain map $0 : C \to 0$ is a quasi-isomorphism, that is $\mathrm{H}_n(0)$ is an isomorphism for every $n \in \mathbb{Z}$.

Exercise 2

In this exercise we practise to take a categorical perspective on homological algebra.

(i) Let (C, d) be a chain complex. Provide a definition of $\mathrm{H}_n(C)$ using the categorial notions of kernel and cokernel.

(ii) Let $\varphi : C \to D$ be a chain map. Prove that $\mathrm{H}_n(\varphi)$ is well-defined using the categorical definition of $\mathrm{H}_n(C)$.

Exercise 3 (to be corrected)

In this exercise we introduce the Euler characteristic of a chain complex. The exercises conclusion admits an interpretation in the context of Δ-complexes, their Euler characteristic and their simplicial homology.

A chain complex C is bounded if there are $n_l < n_u$ such that $C_n = 0$ for all $n < n_l$ or $n_u < n$. The Euler characteristic of a bounded chain complex C of free modules of finite rank is defined as $\chi(C) = \sum_{n \in \mathbb{Z}} (-1)^n \mathrm{rank}(C_n)$. The rank of an arbitrary abelian group A is defined as $\mathrm{rank}\, A = \dim_{\mathbb{Q}} \mathbb{Q} \otimes_{\mathbb{Z}} A$.

Assume that C is a bounded chain complex of free $\mathbb{Z}$-modules. Show that

$$\chi(C) = \sum_{n \in \mathbb{Z}} (-1)^n \operatorname{rank} \mathrm{H}_n(C)\,.$$

Exercise 4 (to be corrected)

Fix the model of $\mathrm{S}^n = \{t \in \mathbb{R}^{n+1} \mid \|t\|_2 = 1\}$. The antipode $\varphi : \mathrm{S}^n \to \mathrm{S}^n$ is defined by $\varphi(t) = -t$. Note that $\varphi^2 = \mathrm{id}_{\mathrm{S}^n}$, that $\mathbb{Z}_2$ acts on S^n by the antipode. The real projective space is defined as $\mathbb{RP}^n = \mathrm{S}^n / \mathbb{Z}_2$.

(i) Find a Δ-complex structure on S^n which is compatible with the antipode, that is $(\sigma_\alpha)_{\alpha \in A}$ with simplices $\sigma_\alpha : \Delta^{n(\alpha)} \to \mathrm{S}^n$ such that for all $\alpha \in A$ there is $\beta \in A$ satisfying $\varphi \circ \sigma_\alpha = \sigma_\beta$.

(ii) Construct a Δ-complex structure on $\mathbb{RP}^n$.

(iii) Calculate the Euler characteristic of $\mathbb{RP}^n$.