Exercises on Homology and Cohomology

Spring term 2018, Sheet 3

Hand in before 10 o'clock on 12th March 2018
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Exercise 1

Let X be a finite Δ-complex, that is a Δ-complex with only finitely many simplices. Show that

$$\chi(X) = \sum_{n=0}^{\infty} (-1)^n \operatorname{rank} \mathrm{H}_n^{\Delta}(X) \,.$$

Exercise 2 (to be corrected)

The dimension of a Δ-complex is the largest dimension of a simplex it contains. Let X and Y be a connected finite Δ-complexes of the same finite dimension n. We defined the connected sum $X \# Y$ combinatorially by first choosing injective n-simplices $\sigma_X : \Delta^n \hookrightarrow X$ and $\sigma_Y : \Delta^n \hookrightarrow Y$, removing their interior and finally gluing the remaining parts along the simplices' boundaries. In formulas:

$$X \# Y = \Big((X \setminus (\operatorname{im} \sigma_X)^\circ) \sqcup (Y \setminus (\operatorname{im} \sigma_Y)^\circ) \Big) / \, \sigma_X(t) = \sigma_Y(t) \text{ for all } t \in \partial \Delta^n \,.$$

(i) Calculate the Euler characteristic of $X \# Y$ in terms of the Euler characteristic of X and Y.

(ii) Use the previous formula to calculate the Euler characteristic of the standard surface of genus g, $g \geq 1$, which satisfies

$$\Sigma_g = \mathbb{T}^2 \# \cdots \# \mathbb{T}^2 \quad (g \text{ terms})$$

Exercise 3

The standard surface Σ_g of genus g is obtained as a quotient of the regular $4g$-gon. Recall the Δ-complex structure on Σ_g obtained by fixing some vertex of the $4g$-gon and connecting it with all others.

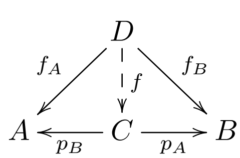
(i) Based on the previous Δ-complex structure, calculate the simplicial homology $\mathrm{H}_n(\Sigma_g)$ of the standard surface of genus g.

(ii) Describe a generator of $\mathrm{H}_2(\Sigma_g)$.

(iii) Check your calculations by matching $\chi(\Sigma_g) = \sum_{n=0}^{\infty} (-1)^n \operatorname{rank} \mathrm{H}_n(\Sigma_g)$.

Exercise 4 (to be corrected)

Let $\mathcal{C}$ be a category and $A, B \in \operatorname{ob} \mathcal{C}$. An object $C \in \operatorname{ob} \mathcal{C}$ is called the product of A, B if there are morphisms $p_A : C \to A$, $p_B : C \to B$ such that for any pair of morphisms $f_A : D \to A$ and $f_B : D \to B$ there is a unique morphism $f : D \to C$ such that the following diagram commutes.



We write $A \prod B$ for the product of A and B.

(i) Show that $A \prod B$, if it exists, is unique up to a unique isomorphism in the following sense: whenever C a product of A and B then there is a unique isomoprhism $A \prod B \to C$ that makes the following diagram commute.

$$\begin{array}{ccccc} A & \xleftarrow{p_B} & A \prod B & \xrightarrow{p_A} & B \\ \downarrow{\scriptstyle \mathrm{id}_A} & & \downarrow{\scriptstyle \cong} & & \downarrow{\scriptstyle \mathrm{id}_B} \\ A & \longleftarrow & C & \longrightarrow & B \end{array}$$

(ii) Show that products exist in the category $\mathbf{Grp}$ of groups and agree with the usual notion of product groups.

(iii) The definition of a coproduct is dual to the definition of a product, in the sense that all arrows in the definition have to be reversed. Define the coproduct of two objects A, B in a category $\mathcal{C}$.

(iv) Does the category $\mathbf{Grp}$ have coproducts? Prove their existence or provide an example of two groups G, H, which do not have a coproduct.