Exercises on Homology and Cohomology

Spring term 2018, Sheet 6

Hand in before 10 o'clock on 9th April 2018
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Exercise 1 (easy)

Let $\varphi : C \to C$ be a endomorphism of a chain complex of R-modules that is chain homotopic to the identity, say there is an R-module map $h : C_* \to C_{*+1}$ satisfying $\partial h + h\partial = \mathrm{id}_C - \varphi$. Show that for all $n \in \mathbb{N}_{\geq 1}$ the n-fold iteration φ^n is homotopic with id_C via the map $\sum_{0 \leq i < n} h\varphi^i$.

Exercise 2 (medium)

Prove the Five Lemma: given a commutative diagram of R-modules with exact rows such that f and h are isomorphisms, then g is an isomorphism too.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A' & \longrightarrow & B' & \longrightarrow & C' & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle h} & & \\
0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0
\end{array}$$

Exercise 3 (difficult)

The complex projective space of dimension n is the space of one-dimensional subspaces of $\mathbb{C}^{n+1}$ modelled as the quotient space of the unit sphere in $\mathbb{C}^{n+1}$ by the diagonal multiplication action $\mathrm{S}^1 \curvearrowright \mathbb{C}^{n+1}$ defined by $\lambda(x_0, \ldots, x_n) = (\lambda x_0, \ldots, \lambda x_n)$. We define $\mathbb{C}\mathrm{P}^n \stackrel{\text{def}}{=} \{x \in \mathbb{C}^{n+1} \mid \|x\| = 1\}/\mathrm{S}^1$.

(i) Provide a decomposition $\mathbb{C}\mathrm{P}^n = \mathbb{C}\mathrm{P}^{n-1} \cup \mathbb{C}^n$.

(ii) Show that $\mathbb{C}\mathrm{P}^n \setminus \{\mathrm{pt}\}$ is homotopy equivalent with $\mathbb{C}\mathrm{P}^{n-1}$.

(iii) Let H_* be a homology theory in the sense of Eilenberg-Steenrod with coefficient group A. Compute $\mathrm{H}_*(\mathbb{C}P^n)$.