Exercises on Lie groups

Spring term 2018, Sheet 1

Hand in before 10 o'clock on 23rd February 2018 — Sven Raum
Mail box of Sven Raum in MA B2 475 — Gabriel Jean Favre

Notice

There will be a crash course in smooth manifolds offered by Davide Parise, which might be helpful for participants who do not have a differential geometry background. Precise dates are not yet known, but it will probably start from next week on.

Exercise 1

For each classical group, decide with a proof whether it is compact or not.

Exercise 2

Fix $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$.

(i) Show that $\det(A) = 1$ for all $A \in \mathrm{Sp}(n, \mathbb{K})$.

(ii) Conclude that $\mathrm{Sp}(n, \mathbb{K}) \subset \mathrm{SL}(n, \mathbb{K})$. For which $n \in \mathbb{N}_{\geq 1}$ is this inclusion an equality?

Exercise 3

This exercise provides another construction of the group $\mathrm{Sp}(n)$ as a group of quaterionic unitaries. Let $\mathbb{H} = \{a + bi + cj + dk \mid a, b, c, d \in \mathbb{R}\}$ be the $\mathbb{R}$-algebra of quaternions, defined by $ij = k = -ji$, $jk = i = -kj$, $ki = j = -ik$ and $i^2 = j^2 = k^2 = -1$. Note that the subalgebra generated by $1, i$ is isomorphic with $\mathbb{C}$. Let $\mathbb{H}^n$ be the vector space of n-tuples of elements in $\mathbb{H}$, which becomes a bimodule over $\mathbb{H}$, by left and right multiplication:

$$h(h_1, \ldots, h_n)h' = (hh_1h', \ldots, hh_nh')\,,$$

for all $h, h' \in \mathbb{H}$ and all $(h_1, \ldots, h_n) \in \mathbb{H}^n$.

(i) Show that the algebra of right-$\mathbb{H}$-module endomorphisms $\mathrm{End}_{-\mathbb{H}}(\mathbb{H}^n)$ is isomorphic with $\mathrm{M}_n(\mathbb{H})$ acting by left multiplication. We will make the identification $\mathrm{End}_{-\mathbb{H}}(\mathbb{H}^n) \cong \mathrm{M}_n(\mathbb{H})$ what follows.

(ii) For $h = a + bi + cj + dk \in \mathbb{H}$ let $\overline{h} = a - bi - cj - dk$. Show that

$$\langle h, h' \rangle = \sum_{i=1}^{n} h_i \overline{h'_i}$$

for $h = (h_1, \ldots, h_n), h' = (h'_1, \ldots, h'_n) \in \mathbb{H}^n$ defines an $\mathbb{H}$-valued $\mathbb{R}$-bilinear form on $\mathbb{H}^n$.

(iii) We define the group of "unitary quaternionic transoformations" as

$$\mathrm{U}(n, \mathbb{H}) = \{A \in \mathrm{M}_n(\mathbb{H}) \mid \forall h, h' \in \mathbb{H}^n : \langle Ah, Ah' \rangle = \langle h, h' \rangle\}\,.$$

Show that $\mathrm{U}(n, \mathbb{H})$ is indeed a group and that

$$\mathrm{U}(n, \mathbb{H}) = \{A \in \mathrm{M}_n(\mathbb{H}) \mid A^*A = 1\}\,,$$

where $(A^*)_{ij} = (\overline{A_{ji}})$.

(iv) Check that $\mathbb{H}^n$ is a complex vector space when equipped with the scalar multiplication

$$z(h_1, \dots, h_n) = (zh_1, \dots, zh_n)$$

for $z \in \mathbb{C}$ and $(h_1, \dots, h_n) \in \mathbb{H}^n$. Further, show that the map

$$V : \mathbb{C}^{2n} \to \mathbb{H}^n : (z_1, \dots, z_{2n}) \mapsto (z_1 + z_{n+1}j, \dots, z_n + z_{2n}j)$$

is an isomorphism of complex vector spaces.

(v) Show that

$$\langle Vz, Vz' \rangle = \langle z, z' \rangle - (z, z')j$$

for all $z, z' \in \mathbb{C}^{2n}$, where $\langle z, z' \rangle = \sum_{i=1}^{2n} z_i \overline{z_i'}$ is the standard scalar product on $\mathbb{C}^{2n}$ and $(z, z') = \sum_{i=1}^{n} z_{n+i} z_i' - z_i z_{n+i}' = \langle Jz, z' \rangle_{\mathbb{R}}$ is the standard skew symmetric form on $\mathbb{C}^{2n}$

$$J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix} \in \mathrm{M}_{2n}(\mathbb{R}).$$

(vi) Conclude that $V^{-1}\mathrm{U}(n, \mathbb{H})V = \mathrm{Sp}(n, \mathbb{C}) \cap \mathrm{U}(2n) = \mathrm{Sp}(n)$. (See Exercise 2 for the last equality).

Exercise 4

This exercise verifies that classical groups are Lie groups. A topological group is a group G equipped with a topology such that multiplication and inversion are continuous maps. Based on this notion, one equivalent way to define Lie groups is as follows: a Lie group is a topological group whose underlying space is a topological manifold.

Prove that all classical Lie groups are Lie groups in the sense of this definition.

Exercise 5

This exercise observes an important property of connected groups. At the same time it clarifies the role of countability axioms in the (to be given) definition of Lie groups. A topological group is a group G equipped with a topology such that multiplication and inversion are continuous maps.

(i) Assume that G is a connected topological group and let $U \subset G$ be a neighbourhood of the neutral element. Show that U generates G as an abstract group.

(ii) Let G be a connected topological group, whose underlying space is a topological manifold. Show that the topology of G is first countable.