Exercises on Lie groups

Spring term 2018, Sheet 3

Hand in before 10 o'clock on 9th March 2018
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Exercise 1

The global rank theorem from differential geometry says that any bijective differential map of constant rank between differential manifolds is a diffeomorphism.

(i) Show that every Lie group homomorphism has constant rank.

(ii) Deduce from the previous statement and the global rank theorem that a Lie group homomorphism is a Lie group isomorphism if and only if it is bijective.

Exercise 2.

Let G be a Lie group with Lie algebra $(G) = \mathfrak{g}$ and multiplication $m : G \times G \to G$ and inversion $i : G \to G$. Identify $\mathfrak{g} \cong \mathcal{D}^1(G, e)$.

(i) Calculate the differential $(\mathrm{d}m)_e : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$

(ii) Calculate the differential $(\mathrm{d}i)_e : \mathfrak{g} \to \mathfrak{g}$.

Exercise 3.

(i) Let $A \subset X$ be an inclusion of topological spaces. Show that the following statements are equivalent.

  (a) $A \subset X$ is locally closed, that is for all $x \in A$ there is some open neighbourhood $x \in U \subset X$ such that $A \cap U \subset U$ is closed.
  (b) $A \subset \overline{A}$ is open.
  (c) There is a closed subset $C \subset X$ and an open subset $U \subset X$ such that $A = C \cap U$.

(ii) Let G be a Lie group and $H \leq G$ a Lie subgroup. Use the previous item to show that the following statements are equivalent.

  (a) $H \leq G$ is a closed subgroup.
  (b) $H \leq G$ is an embedded submanifold, that is H carries the subspace topology of G.

Exercise 4.

(i) Show that $\mathbb{R}^3$ with cross product is a Lie algebra. Denote this Lie algebra by $\mathfrak{g}$.

(ii) Find a Lie group G whose Lie algebra is $\mathrm{Lie}(G) = \mathfrak{g}$.