Exercises on Lie groups

Spring term 2018, Sheet 5

Hand in before 10 o'clock on 23th March 2018
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Exercise 1.
Let $A, B \in \mathrm{M}_n(\mathbb{K})$.

(i) Show that if A, B commute, then $\exp(A+B) = \exp(A)\exp(B)$.

(ii) Show that if $\exp(A)$ and $\exp(B)$ commute and lie inside the unit ball around the identity matrix, then also A and B commute.

Exercise 2.
Let G be a Lie group and $\mathfrak{g}$ its Lie algebra.

(i) Show that $\mathrm{Aut}(\mathfrak{g})$ is a matrix Lie group.

(ii) Denote by $\mathrm{Der}(\mathfrak{g})$ the algebra of derivations of $\mathfrak{g}$, that is the set of all linear maps $D : \mathfrak{g} \to \mathfrak{g}$ such that $D([X,Y]) = [D(X), Y] + [X, D(Y)]$ holds for all $X, Y \in \mathfrak{g}$. Show that $\mathrm{Der}(\mathfrak{g}) \cong \mathrm{Lie}(\mathrm{Aut}(\mathfrak{g}))$.

(iii) Denote by $\mathrm{Ad}\, g : \mathfrak{g} \to \mathfrak{g}$ the derivative of the adjoint $\mathrm{Ad}\, g : G \to G$ and consider the Lie group homomorphism $\mathrm{Ad} : G \to \mathrm{Aut}(\mathfrak{g})$. Show that the derivative of Ad is the map $\mathrm{ad} : \mathfrak{g} \to \mathrm{Der}(\mathfrak{g})$ given by $(\mathrm{ad}\, X)(Y) = [X, Y]$.

Exercise 3.
Show that the Lie algebra of $\mathrm{Sp}(n, \mathbb{K})$ is

$$\mathfrak{sp}(n, \mathbb{K}) = \left\{ A \in \mathfrak{gl}(n, \mathbb{K}) \mid A^{\mathrm{t}} \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix} = -\begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix} A \right\}$$

Exercise 4.
Let $a, b, c \in \mathbb{R}$ and define elements in $\mathfrak{gl}(3, \mathbb{R})$ by

$$A = \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix} \quad B = \begin{pmatrix} 0 & a & c \\ 0 & 0 & b \\ 0 & 0 & 0 \end{pmatrix} \quad C = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Find explicit formulas for the one-parameter subgroups of $\mathrm{GL}(3, \mathbb{R})$ generated by A, B, C.

Exercise 5.
Let $H \leq G$ be a Lie subgroup. Show that $G = \coprod_{g \in G/H} gH$ are the leaves of a foliation of G.