Exercises on Lie groups

Spring term 2018, Sheet 8

Hand in before 10 o'clock on 27th April 2018
Mail box of Sven Raum in MA B2 475

Sven Raum
Gabriel Jean Favre

Exercise 1

In this exercise we investigate the fundamental group of Lie groups. Recall from algebraic topology that if (X, x) is a pointed topological space, then its fundamental group $\Gamma = \pi_1(X, x)$ is a discrete group acting on the universal cover $\Gamma \curvearrowright (\tilde{X}, \tilde{x})$ such that the quotient map $\tilde{X} \to \tilde{X}/\Gamma \cong X$ is the universal covering map.

(i) Let $H \trianglelefteq G$ be a normal closed subgroup of a connected Lie group. Show that the projection $G \to G/H$ is a local isomorphism if and only if H is discrete in G.

(ii) Show that every discrete normal subgroup of a connected Lie group is central.

(iii) If G is a connected Lie group, show that there is a natural embedding $\pi_1(G) \to \tilde{G}$.

(iv) Conclude that the fundamental group of a connected Lie group is commutative.

Exercise 2 (Exercise 2.5.9 from the course).

Let $(X_1, x_1), (X_2, x_2)$ be pointed topological spaces and prove the following statements.

(i) If $(Y_i, y_i) \to (X_i, x_i)$ is a covering for $i \in \{1, 2\}$, then also

$$(Y_1 \times Y_2, (y_1, y_2)) \to (X_1 \times X_2, (x_1, x_2))$$

is a covering map.

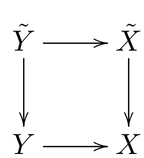
(ii) If there are universal coverings $(\tilde{X}_i, \tilde{x}_i) \to (X_i, x_i)$ for $i \in \{1, 2\}$, then

$$(\tilde{X}_1 \times \tilde{X}_2, (\tilde{x}_1, \tilde{x}_2)) \to (X_1 \times X_2, (x_1, x_2))$$

is a universal covering

Exercise 3 (Exercise 2.5.10 from the course).

Let Y, X be pointed topological spaces admitting a universal cover. Let $Y \to X$ be a pointed map. Show that there is a unique pointed map $\tilde{Y} \to \tilde{X}$ such that the following diagram commutes



Exercise 4.
In this exercise we calculate the fundamental group of the Lie group $\mathrm{SO}(n,\mathbb{R})$. It makes use of methods from advanced algebraic topology.

(i) Show that for $n \geq 2$, we have $\mathrm{SO}(n,\mathbb{R})/\mathrm{SO}(n-1,\mathbb{R}) \cong \mathrm{S}^{n-1}$ and that the translation action

$$\mathrm{SO}(n,\mathbb{R}) \curvearrowright \mathrm{SO}(n,\mathbb{R})/\mathrm{SO}(n-1,\mathbb{R})$$

is conjugate to the standard action $\mathrm{SO}(n,\mathbb{R}) \curvearrowright \mathrm{S}^{n-1}$.

(ii) Show inductively that $\mathrm{SO}(n,\mathbb{R})$ is (path) connected for every $n \in \mathbb{N}_{>0}$.

(iii) Show that $\mathrm{SO}(n,\mathbb{R})$ is an $\mathrm{SO}(n-1,\mathbb{R})$-principal bundle over S^{n-1}, that is fixing the projection $p : \mathrm{SO}(n,\mathbb{R}) \to \mathrm{S}^{n-1}$, for every $x \in \mathrm{S}^{n-1}$ there is a neighbourhood $x \in U \subset \mathrm{S}^{n-1}$ and an $\mathrm{SO}(n-1,\mathbb{R})$-equivariant isomorphism

$$\begin{array}{ccc} U \times \mathrm{SO}(n-1,\mathbb{R}) & \xrightarrow{\cong} & p^{-1}(U) \\ \downarrow{\scriptstyle \pi_U} & & \downarrow{\scriptstyle p} \\ U & \xrightarrow{\mathrm{id}} & U \end{array}$$

(iv) From a principal, one obtains a short exact sequence of homotopy groups. Use the short exact sequence of homotopy groups obtained from this previous bundle

$$\pi_2(\mathrm{S}^{n-1}) \to \pi_1(\mathrm{SO}(n-1,\mathbb{R})) \to \pi_1(\mathrm{SO}(n,\mathbb{R})) \to \pi_1(\mathrm{S}^{n-1}) \to 1$$

in order to calculate $\pi_1(\mathrm{SO}(n,\mathbb{R}))$ inductively. Use the fact that $\mathrm{SO}(3,\mathbb{R}) \cong \mathbb{R}\mathrm{P}^3$.