Exercises on Lie groups

Spring term 2018, Sheet 9

Hand in before 10 o'clock on 4th May 2018
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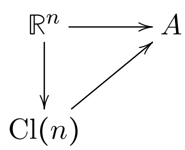
Exercise 1

In this exercise we study the universal cover of $\mathrm{SO}(n)$, which is the so-called spin group $\mathrm{Spin}(n)$.

(i) Denote by $\mathrm{T}\mathbb{R}^n$ the tensor algebra of $\mathbb{R}^n$. We define

$$\mathrm{Cl}(n) \coloneqq \mathrm{T}\mathbb{R}^n / v \otimes v - \|v\|^2 .$$

Show that map $\mathbb{R}^n \to \mathrm{Cl}(n)$ is injective and that $\mathrm{Cl}(n)$ enjoys the following universal property: whenever $i : \mathbb{R}^n \to A$ is a vector space homomorphism into a unital algebra such that $i(v)^2 = \|v\|^2 1$, then there is a unique unital homomorphism $\mathrm{Cl}(n) \to A$ making the following diagram commutative.



(ii) Show that for the elements from the standard basis $e_1, \dots, e_n \in \mathbb{R}^n$ the following relation holds in $\mathrm{Cl}(n)$.

$$e_i e_j = \begin{cases} -e_j e_i & i \neq j \\ e_j e_i & i = j . \end{cases}$$

(iii) Show that $\mathrm{Cl}(n)$ is finite dimensional.

(iv) Let

$$\mathrm{Pin}(n) = \{ v_{i_1} \cdots v_{i_k} \in \mathrm{Cl}(n) \mid v_{i_1}, \dots, v_{i_k} \in \mathbb{R}^n \text{ and } \|v_{i_1}\| = \cdots \|v_{i_k}\| = 1 \}$$

Show that $\mathrm{Pin}(n)$ is the group with the multiplication inherited from $\mathrm{Cl}(n)$.

(v) Denote by $\mathrm{Cl}^{\mathrm{even}}(n) = \bigcup_{k \in \mathbb{N}} \mathrm{Cl}^{2k}(n)$ the even part of $\mathrm{Cl}(n)$. Show that $\mathrm{Cl}^{\mathrm{even}}(n) \leq \mathrm{Cl}(n)$ is a subalgebra.

(vi) Define the spin group as

$$\mathrm{Spin}(n) = \mathrm{Pin}(n) \cap \mathrm{Cl}^{\mathrm{even}}$$

and show that it is a connected Lie group.

(vii) Show that $v_1 \cdots v_k \mapsto (v_1 \cdots v_k)^{\mathrm{t}} = v_k \cdots v_1$ defines an anti-automorphism of $\mathrm{Cl}(n)$, that is $(xy)^{\mathrm{t}} = y^{\mathrm{t}} x^{\mathrm{t}}$.

(viii) Show that for formula $\rho(g)v \coloneqq g v g^{\mathrm{t}}$ defines a smooth action of $\mathrm{Spin}(n)$ on $\mathbb{R}^n = \mathrm{Cl}^1(n)$ and the induced homomorphism of Lie groups $\mathrm{Spin}(n) \to \mathrm{SO}(n)$ is the universal cover.