Homology and Cohomology

Spring term 2018, Trial exam

Time: 3 hours
Max 90 points

Sven Raum
Haoqing Wu

Name: ______________________________

Sciper: ______________________________

General explanations

- The exam takes 3 hours, or equivalently 180 minutes.
- You can achieve a maximum of 90 points, while points for all questions sum up to 115.

Question 1 (Euler characteristic, 20 points)
In this question X denotes a topological space with a finite Δ-complex structure and $\mathrm{S}^n = \{t \in \mathbb{R}^{n+1} \mid \|t\|_2 = 1\}$ denotes the n-sphere.

(i) Give the definition of the Euler characteristic $\chi(X)$ of a finite Δ-complex X.

(ii) Show that $\chi(X) = \sum_{i=0}^{\infty} \operatorname{rank} \mathrm{H}_i(X)$, where H_i denotes singular homology of X.

(iii) Calculate $\chi(\mathrm{S}^n)$.

Question 2 (Eilenberg-Steenrod axioms, 15 points)

(i) Give the definition of a homology theory in the sense of Eilenberg-Steenrod.

(ii) Prove that singular homology is additive.

Question 3 (Cellular homology of surfaces, 30 points)
In this question Σ_2 denotes the standard surface of genus 2, modelled as the quotient of a regular 8-gon by the identification of its edges $a, b, a^{-1}, b^{-1}, c, d, c^{-1}, d^{-1}$.

(i) Let X be a CW-complex. Give the definition of the cellular chain complex of X, including its boundary maps.

(ii) Find a CW-complex structure on Σ_2. Hint: first construct a CW-structure on the regular 8-gon and derive from this a CW-complex structure on Σ_2.

(iii) Calculate cellular homology of Σ_2 with respect to the CW-complex structure exhibited in (ii).

Question 4 (Projective resolutions, 30 points)
In this question R denotes an arbitrary ring and M is an arbitrary M-module.

(i) Give the definition of a projective module over R.

(ii) Give the definition of a projective resolution of M over R.

(iii) Show that there is a unique projective resolution of M over R up to chain homotopy equivalence.

Question 5 (Group cohomology, 20 points)
In this question $G = \mathbb{Z}/2\mathbb{Z}$ denotes the cyclic group of order two and M denotes an arbitrary $\mathbb{Z}G$-module.

(i) Find a free resolution F_* of the trivial $\mathbb{Z}G$-module $\mathbb{Z}$, satisfying $F_k \cong \mathbb{Z}G$ for all $k \in \mathbb{N}$.

(ii) Calculate group cohomology $\mathrm{H}^*(\mathbb{Z}/2\mathbb{Z}, M)$ in terms of invariants and coinvariants of M.